

# Rigidity of proper holomorphic mappings between equidimensional Hua domains

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**Abstract.** Hua domain, named after Chinese mathematician Loo-Keng Hua, is defined as a domain in  $\mathbb{C}^n$  fibered over an irreducible bounded symmetric domain  $\Omega \subset \mathbb{C}^d$  ( $d < n$ ) with the fiber over  $z \in \Omega$  being a  $(n - d)$ -dimensional generalized complex ellipsoid  $\Sigma(z)$ . In general, a Hua domain is a nonhomogeneous domain without smooth boundary. The purpose of this paper is twofold. Firstly, we obtain what seems to be the first rigidity results on proper holomorphic mappings between two equidimensional Hua domains. Secondly, we determine the explicit form of the biholomorphisms between two equidimensional Hua domains. As a special conclusion of this paper, we completely describe the group of holomorphic automorphisms of the Hua domain.

**Key words:** Bounded symmetric domains, Generalized complex ellipsoids, Holomorphic automorphisms, Hua domains, Proper holomorphic mappings

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## 1 Introduction

Before we introduce Hua domains, we first recall the results on generalized complex ellipsoids and bounded symmetric domains. A generalized complex ellipsoid (also called generalized pseudoellipsoid) is a domain of the form

$$\Sigma(\mathbf{n}; \mathbf{p}) = \left\{ (\zeta_1, \dots, \zeta_r) \in \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \sum_{k=1}^r \|\zeta_k\|^{2p_k} < 1 \right\},$$

where  $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ ,  $\mathbf{p} = (p_1, \dots, p_r) \in (\mathbb{R}_+)^r$ , and  $\|\cdot\|$  is the standard Hermitian norm. By relabelling the coordinates, we can always assume that  $p_2 \neq 1, \dots, p_r \neq 1$ , that is, there is at most one 1 in  $p_1, \dots, p_r$ .

In the special case where all the  $p_k = 1$ , the generalized complex ellipsoid  $\Sigma(\mathbf{n}; \mathbf{p})$  reduces to the unit ball in  $\mathbb{C}^{n_1 + \dots + n_r}$ . Also, it is known that a generalized complex ellipsoid  $\Sigma(\mathbf{n}; \mathbf{p})$  is homogeneous if and only if  $p_k = 1$  for all  $1 \leq k \leq r$  (cf. Kodama [14]). In general, a generalized complex ellipsoid is not strongly pseudoconvex and its boundary is not smooth.

For the biholomorphic mappings between two equidimensional generalized complex ellipsoids, in 1968, Naruki [21] proved the following result.

**Theorem 1.A** (Naruki [21]) *Let  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$  be two equidimensional generalized complex ellipsoids with  $\mathbf{n}, \mathbf{m} \in \mathbb{N}^r$  and  $\mathbf{p}, \mathbf{q} \in (\mathbb{R}_+)^r$  (where  $p_k \neq 1, q_k \neq 1$  for  $2 \leq k \leq r$ ). Then  $\Sigma(\mathbf{n}; \mathbf{p})$  is biholomorphic to  $\Sigma(\mathbf{m}; \mathbf{q})$  if and only if there exists a permutation  $\sigma \in S_r$  (where  $S_r$  is the permutation group of the  $r$  numbers  $\{1, \dots, r\}$ ) such that  $n_{\sigma(j)} = m_j, p_{\sigma(j)} = q_j$  for  $1 \leq j \leq r$ .*

The holomorphic automorphism group  $\text{Aut}(\Sigma(\mathbf{n}; \mathbf{p}))$  of  $\Sigma(\mathbf{n}; \mathbf{p})$  has been studied by Dini-Primicerio [7], Kodama [14] and Kodama-Krantz-Ma [15]. In 2013, Kodama [14] obtained the result as follows.

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**Theorem 1.B** (Kodama [14]) (i) *If 1 does not appear in  $p_1, \dots, p_r$ , then any automorphism  $\varphi \in \text{Aut}(\Sigma(\mathbf{n}; \mathbf{p}))$  is of the form*

$$\varphi(\zeta_1, \dots, \zeta_r) = (\gamma_1(\zeta_{\sigma(1)}), \dots, \gamma_r(\zeta_{\sigma(r)})), \quad (1)$$

where  $\sigma \in S_r$  is a permutation of the  $r$  numbers  $\{1, \dots, r\}$  such that  $n_{\sigma(i)} = n_i, p_{\sigma(i)} = p_i$  ( $1 \leq i \leq r$ ) and  $\gamma_1, \dots, \gamma_r$  are unitary transformations of  $\mathbb{C}^{n_1}(n_{\sigma(1)} = n_1), \dots, \mathbb{C}^{n_r}(n_{\sigma(r)} = n_r)$  respectively.

(ii) *If 1 appears in  $p_1, \dots, p_r$ , we can assume, without loss of generality, that  $p_1 = 1, p_2 \neq 1, \dots, p_r \neq 1$ , then  $\text{Aut}(\Sigma(\mathbf{n}; \mathbf{p}))$  is generated by elements of the form (1) and automorphisms of the form*

$$\varphi_a(\zeta_1, \zeta_2, \dots, \zeta_r) = \left( T_a(\zeta_1), \zeta_2(\psi_a(\zeta_1))^{\frac{1}{2p_2}}, \dots, \zeta_r(\psi_a(\zeta_1))^{\frac{1}{2p_r}} \right), \quad (2)$$

where  $T_a$  is an automorphism of the ball  $\mathbf{B}^{n_1}$  in  $\mathbb{C}^{n_1}$ , which brings a point  $a \in \mathbf{B}^{n_1}$  in the origin and

$$\psi_a(\zeta_1) = \frac{1 - \|a\|^2}{(1 - \langle \zeta_1, a \rangle)^2}.$$

Every bounded symmetric domain is, when equipped with the Bergman metric, a Hermitian symmetric manifold of noncompact type, and every Hermitian symmetric manifold of noncompact type can be realized as a bounded symmetric domain in some  $\mathbb{C}^d$  by the Harish-Chandra embedding theorem. In 1935, E. Cartan proved that there exist only six types of irreducible bounded symmetric domains. They are four types of classical bounded symmetric domains and two exceptional domains. So bounded symmetric domains are also known as Cartan domains.

Let  $\Omega$  be an irreducible bounded symmetric domain in  $\mathbb{C}^d$  of genus  $g$  in its Harish-Chandra realization. Let

$$\left\{ \frac{1}{\sqrt{V(\Omega)}}, h_1(z), h_2(z), \dots \right\}$$

be an orthonormal basis of the Hilbert space  $A^2(\Omega)$  of square-integrable holomorphic functions on  $\Omega$ . Define the Bergman kernel  $K_\Omega(z, \bar{\xi})$  of  $\Omega$  by

$$K_\Omega(z, \bar{\xi}) := \frac{1}{V(\Omega)} + \sum_{i=1}^{\infty} h_i(z) \overline{h_i(\xi)}$$

for all  $z, \xi \in \Omega$ . Obviously,  $1 \leq V(\Omega)K_\Omega(z, \bar{z}) < +\infty$ . The generic norm of  $\Omega$  is defined by

$$N_\Omega(z, \bar{\xi}) := (V(\Omega)K_\Omega(z, \bar{\xi}))^{-\frac{1}{g}} \quad (z, \xi \in \Omega),$$

where  $(V(\Omega)K_\Omega(z, \bar{\xi}))^{-\frac{1}{g}} := \exp(-\frac{1}{g} \log(V(\Omega)K_\Omega(z, \bar{\xi})))$ , in which  $\log$  denotes the principal branch of logarithm (note  $K_\Omega(z, \bar{\xi}) \neq 0$  for all  $z, \xi \in \Omega$ ). Thus  $0 < N_\Omega(z, \bar{z}) \leq 1$  for all  $z \in \Omega$  and  $N_\Omega(z, \bar{z}) = 0$  on the boundary of  $\Omega$ .

For an irreducible bounded symmetric domain  $\Omega \subset \mathbb{C}^d$  in its Harish-Chandra realization, a positive integer  $r$  and  $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ ,  $\mathbf{p} = (p_1, \dots, p_r) \in (\mathbb{R}_+)^r$ , the Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p})$  is defined by

$$\begin{aligned} H_\Omega(\mathbf{n}; \mathbf{p}) &= H_\Omega(n_1, \dots, n_r; p_1, \dots, p_r) \\ &:= \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in \Omega \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \sum_{j=1}^r \|w_{(j)}\|^{2p_j} < N_\Omega(z, \bar{z}) \right\}, \end{aligned}$$

where  $\|\cdot\|$  is the standard Hermitian norm. Note that  $\Omega \times \{0\} \subset H_\Omega(\mathbf{n}; \mathbf{p})$  and  $b\Omega \times \{0\} \subset bH_\Omega(\mathbf{n}; \mathbf{p})$  (where  $bD$  denotes the boundary of a domain  $D$ ).

For  $(z, w_{(1)}, \dots, w_{(r)}) \in H_\Omega(\mathbf{n}; \mathbf{p})$ , by definition, we have

$$\frac{\sum_{j=1}^r \|w_{(j)}\|^{2p_j}}{N_\Omega(z, \bar{z})} = \exp \left\{ \log \left( \sum_{j=1}^r \|w_{(j)}\|^{2p_j} \right) + \frac{1}{g} \log(V(\Omega)K_\Omega(z, \bar{z})) \right\}.$$

Because  $\log \left( \sum_{j=1}^r \|w_{(j)}\|^{2p_j} \right) + \frac{1}{g} \log(V(\Omega)K_\Omega(z, \bar{z}))$  is a plurisubharmonic function on  $H_\Omega(\mathbf{n}; \mathbf{p})$ , we have  $\sum_{j=1}^r \|w_{(j)}\|^{2p_j} / N_\Omega(z, \bar{z})$  is a continuous plurisubharmonic function on  $H_\Omega(\mathbf{n}; \mathbf{p})$ . Since  $\frac{1}{1-x}$  is a monotonically increasing convex function for  $x \in (-\infty, 1)$  and  $0 \leq \sum_{j=1}^r \|w_{(j)}\|^{2p_j} / N_\Omega(z, \bar{z}) < 1$  on  $H_\Omega(\mathbf{n}; \mathbf{p})$ , we have

$$\frac{1}{1 - \sum_{j=1}^r \|w_{(j)}\|^{2p_j} / N_\Omega(z, \bar{z})} = \frac{N_\Omega(z, \bar{z})}{N_\Omega(z, \bar{z}) - \sum_{j=1}^r \|w_{(j)}\|^{2p_j}}$$

is a continuous plurisubharmonic function on  $H_\Omega(\mathbf{n}; \mathbf{p})$ . Thus

$$\max \left\{ \frac{N_\Omega(z, \bar{z})}{N_\Omega(z, \bar{z}) - \sum_{j=1}^r \|w_{(j)}\|^{2p_j}}, \frac{1}{N_\Omega(z, \bar{z})} \right\}$$

is a continuous plurisubharmonic exhaustion function of  $H_\Omega(\mathbf{n}; \mathbf{p})$ . Then  $H_\Omega(\mathbf{n}; \mathbf{p})$  is a bounded pseudoconvex domain in  $\mathbb{C}^{d+n_1+\dots+n_r}$ . But, in general, a Hua domain is a nonhomogeneous domain without smooth boundary.

Let  $\mathcal{M}_{m,n}$  be the set of all  $m \times n$  matrices  $z = (z_{ij})$  with complex entries. Let  $\bar{z}$  be the complex conjugate of the matrix  $z$  and let  $z^t$  be the transpose of the matrix  $z$ .  $I$  denotes the identity matrix. If a square matrix  $z$  is positive definite, then we write  $z > 0$ . For each bounded classical symmetric domain  $\Omega$  (refer to Hua [12]), we list the genus  $g(\Omega)$ , the generic norm  $N_\Omega(z, \bar{z})$  of  $\Omega$  and corresponding Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p})$  (see Yin-Wang-Zhao-Zhao-Guan [32]) according to its type as following.

(i) If  $\Omega = \Omega_I(m, n) := \{z \in \mathcal{M}_{m,n} : I - z\bar{z}^t > 0\} \subset \mathbb{C}^d$  ( $1 \leq m \leq n$ ,  $d = mn$ ) (the classical domains of type  $I$ ), then  $g(\Omega) = m + n$ ,  $N_\Omega(z, \bar{z}) = \det(I - z\bar{z}^t)$ , and

$$H_\Omega(\mathbf{n}; \mathbf{p}) = \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in \Omega_I(m, n) \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \sum_{j=1}^r \|w_{(j)}\|^{2p_j} < \det(I - z\bar{z}^t) \right\}.$$

Specially, when  $\Omega = \mathbf{B}^d$  is the unit ball in  $\mathbb{C}^d$ , then  $N_\Omega(z, \bar{z}) = 1 - \|z\|^2$ , and

$$H_\Omega(\mathbf{n}; \mathbf{p}) = \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in B^d \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \|z\|^2 + \sum_{j=1}^r \|w_{(j)}\|^{2p_j} < 1 \right\}.$$

Thus the Hua domain  $H_{\mathbf{B}^d}(\mathbf{n}; \mathbf{p})$  is just the generalized complex ellipsoid  $\Sigma((d, \mathbf{n}); (1, \mathbf{p}))$ .

(ii) If  $\Omega = \Omega_{II}(n) := \{z \in \mathcal{M}_{n,n} : z^t = -z, I - z\bar{z}^t > 0\} \subset \mathbb{C}^d$  ( $n \geq 2$ ,  $d = n(n-1)/2$ ) (the classical domains of type  $II$ ), then  $g(\Omega) = 2(n-1)$ ,  $N_\Omega(z, \bar{z}) = (\det(I - z\bar{z}^t))^{1/2}$ , and

$$H_\Omega(\mathbf{n}; \mathbf{p}) = \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in \Omega_{II}(n) \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \sum_{j=1}^r \|w_{(j)}\|^{2p_j} < (\det(I - z\bar{z}^t))^{1/2} \right\}.$$

(iii) If  $\Omega = \Omega_{III}(n) := \{z \in \mathcal{M}_{n,n} : z^t = z, I - z\bar{z}^t > 0\} \subset \mathbb{C}^d$  ( $n \geq 2$ ,  $d = n(n+1)/2$ ) (the classical domains of type  $III$ ), then  $g(\Omega) = n+1$ ,  $N_\Omega(z, \bar{z}) = \det(I - z\bar{z}^t)$ , and

$$H_\Omega(\mathbf{n}; \mathbf{p}) = \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in \Omega_{III}(n) \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \sum_{j=1}^r \|w_{(j)}\|^{2p_j} < \det(I - z\bar{z}^t) \right\}.$$

(iv) If  $\Omega = \Omega_{IV}(n) := \{z \in \mathbb{C}^n : 1 - 2z\bar{z}^t + |zz^t|^2 > 0, z\bar{z}^t < 1\}$  ( $n \geq 3$ ) (the classical domains of type IV), then  $g(\Omega) = n$ ,  $N_\Omega(z, \bar{z}) = 1 - 2z\bar{z}^t + |zz^t|^2$ , and

$$H_\Omega(\mathbf{n}; \mathbf{p}) = \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in \Omega_{IV}(n) \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \sum_{j=1}^r \|w_{(j)}\|^{2p_j} < 1 - 2z\bar{z}^t + |zz^t|^2 \right\}.$$

Let  $\Omega$  be an irreducible bounded symmetric domain in  $\mathbb{C}^d$  in its Harish-Chandra realization. We can always assume that the Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p})$  is written *in its standard form*, that is,

(i) If  $\Omega$  is the unit ball, then  $p_1 \neq 1, \dots, p_r \neq 1$  (here it is understood that this domain is the unit ball in  $\mathbb{C}^d$  if  $r = 0$ .);

(ii) If  $\text{rank}(\Omega) \geq 2$ , then  $p_1 = 1, p_2 \neq 1, \dots, p_r \neq 1$  (here it is understood that  $p_1 = 1$  does not appear if  $n_1 = 0$ ).

It is easy to see that every Hua domain can be written *in its standard form* by relabelling the coordinates. Therefore, for every given Hua domain, there exists an irreducible bounded symmetric domain  $\Omega$  in its Harish-Chandra realization such that the Hua domain can be written as  $H_\Omega(\mathbf{n}; \mathbf{p})$  *in its standard form*.

Let  $\Omega$  be an irreducible bounded symmetric domain in  $\mathbb{C}^d$ , and  $\mathbf{n} \in \mathbb{N}^r$ ,  $\mathbf{p} \in (\mathbb{R}_+)^r$ . Let the family  $\Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$  be exactly the set of all mappings  $\Phi$ :

$$\Phi(z, w_{(1)}, \dots, w_{(r)}) = \left( \varphi(z), U_1(w_{(1)}) \frac{(N_\Omega(z_0, \bar{z}_0))^{\frac{1}{2p_1}}}{(N_\Omega(z, \bar{z}_0))^{\frac{1}{p_1}}}, \dots, U_r(w_{(r)}) \frac{(N_\Omega(z_0, \bar{z}_0))^{\frac{1}{2p_r}}}{(N_\Omega(z, \bar{z}_0))^{\frac{1}{p_r}}} \right) \quad (3)$$

for  $(z, w_{(1)}, \dots, w_{(r)}) \in H_\Omega(\mathbf{n}; \mathbf{p})$ , where  $\varphi \in \text{Aut}(\Omega)$ ,  $U_j$  is a unitary transformation of  $\mathbb{C}^{n_j}$  for  $1 \leq j \leq r$ , and  $z_0 = \varphi^{-1}(0)$ . Then  $\Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$  is a subgroup of the holomorphic automorphism group  $\text{Aut}(H_\Omega(\mathbf{n}; \mathbf{p}))$  of  $H_\Omega(\mathbf{n}; \mathbf{p})$  (see Yin-Wang-Zhao-Zhao-Guan [32]). Obviously, every element of  $\Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$  preserves the set  $\Omega \times \{0\} \subset H_\Omega(\mathbf{n}; \mathbf{p})$  and  $\Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$  is transitive on  $\Omega \times \{0\} \subset H_\Omega(\mathbf{n}; \mathbf{p})$ . For the general reference of Hua domains, see Yin-Wang-Zhao-Zhao-Guan [32] and references therein.

When  $r = 1$ , the Hua domain  $H_\Omega(n_1; p_1)$  is also called the Cartan-Hartogs domain and is also denoted by  $\Omega^{B^{n_1}}(p_1)$ . For the reference of the Cartan-Hartogs domains, see Ahn-Byun-Park [1], Feng-Tu [8], Loi-Zedda [17], Wang-Yin-Zhang-Roos [30] and Yin [31] and references therein.

In 2012, Ahn-Byun-Park [1] determined the automorphism group of the Cartan-Hartogs domain  $H_\Omega(n_1; p_1)$  by case-by-case checking only for four types of classical domains  $\Omega$ . Following the reasoning in Ahn-Byun-Park [1], Rong [23] claimed a description of automorphism groups of Hua domains  $H_\Omega(\mathbf{n}; \mathbf{p})$  in 2014. But, Lemma 3.2 in Rong [23], which is central to the proof of its main results in [23], is definitely wrong (cf. Proposition 2.4 in our paper for references).

The first goal of this paper is to give a description of the biholomorphisms between two equidimensional Hua domains. By using a different technique from that in Ahn-Byun-Park [1], we obtain the result as follows.

**Theorem 1.1.** *Suppose that*

$$f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$$

*is a biholomorphism between two equidimensional Hua domains  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  and  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  in their standard forms, where  $\Omega_1 \subset \mathbb{C}^{d_1}$  and  $\Omega_2 \subset \mathbb{C}^{d_2}$  are two irreducible bounded symmetric domains in the Harish-Chandra realization, and  $\mathbf{n}, \mathbf{m} \in \mathbb{N}^r$ ,  $\mathbf{p}, \mathbf{q} \in (\mathbb{R}_+)^r$ . Then there exists an automorphism  $\Phi \in \Gamma(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))$  (see (3) here) and a permutation  $\sigma \in S_r$  with  $n_{\sigma(i)} = m_i, p_{\sigma(i)} = q_i$  for  $1 \leq i \leq r$  such that*

$$\Phi \circ f(z, w_{(1)}, \dots, w_{(r)}) = (z, w_{(\sigma(1))}, \dots, w_{(\sigma(r))}) \begin{pmatrix} A & & & \\ & U_1 & & \\ & & \ddots & \\ & & & U_r \end{pmatrix},$$

where  $A$  is a complex linear isomorphism of  $\mathbb{C}^d$  ( $d := d_1 = d_2$ ) with  $A(\Omega_1) = \Omega_2$ , and  $U_i$  is a unitary transformation of  $\mathbb{C}^{m_i}$  ( $m_i = n_{\sigma(i)}$ ) for  $1 \leq i \leq r$ .

As a special result of Theorem 1.1, we completely describe the automorphism group of the Hua domains  $H_\Omega(\mathbf{n}; \mathbf{p})$  for all irreducible bounded symmetric domains  $\Omega$  as follows.

**Corollary 1.2.** *Let  $H_\Omega(\mathbf{n}; \mathbf{p})$  be a Hua domain in its standard form and  $\Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$  is generated by the mappings of the form (3), where  $\Omega \subset \mathbb{C}^d$  is an irreducible bounded symmetric domain in the Harish-Chandra realization, and  $\mathbf{n} \in \mathbb{N}^r$ ,  $\mathbf{p} \in (\mathbb{R}_+)^r$ . Then, for every  $f \in \text{Aut}(H_\Omega(\mathbf{n}; \mathbf{p}))$ , there exist a  $\Phi \in \Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$  and a permutation  $\sigma \in S_r$  with  $n_{\sigma(i)} = n_i$ ,  $p_{\sigma(i)} = p_i$  for  $1 \leq i \leq r$  such that*

$$f(z, w_{(1)}, \dots, w_{(r)}) = \Phi(z, w_{(\sigma(1))}, \dots, w_{(\sigma(r))}).$$

Remarks on Corollary 1.2. In the case of  $r = 1$ , Ahn-Byun-Park [1] obtained Corollary 1.2 for four types of classical domains  $\Omega$  in 2012. If  $\Omega = \mathbf{B}^d$ , then the Hua domain  $H_{\mathbf{B}^d}(\mathbf{n}, \mathbf{p}) = \Sigma((d, \mathbf{n}), (1, \mathbf{p}))$  is a generalized complex ellipsoid. By (3), we have that Corollary 1.2 implies Theorem 1.B (ii).

It is important that the Hua domain is written *in its standard form* in Corollary 1.2. (i) For example, define

$$H_{\mathbf{B}^2}((2, 2); (1, 2)) = \{(z, w_{(1)}, w_{(2)}) \in \mathbf{B}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 : \|z\|^2 + \|w_{(1)}\|^2 + \|w_{(2)}\|^4 < 1\}.$$

Then, in this case, there exists only the identity  $\sigma = 1 \in S_2$  such that  $n_{\sigma(i)} = n_i$ ,  $p_{\sigma(i)} = p_i$  for  $1 \leq i \leq 2$ , and obviously  $\Gamma(H_{\mathbf{B}^2}((2, 2); (1, 2))) \subsetneq \text{Aut}(H_{\mathbf{B}^2}((2, 2); (1, 2)))$  (cf. Theorem 1.1 in Rong [23]). This means that Corollary 1.2 does not hold for the Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p})$  which is not in the standard form. (ii) If the Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p})$  is the unit ball, then we have  $H_\Omega(\mathbf{n}; \mathbf{p}) = \Omega$  and  $r = 0$  (by  $H_\Omega(\mathbf{n}; \mathbf{p})$  in its standard form). Therefore, we have

$$\Gamma(H_\Omega(\mathbf{n}; \mathbf{p})) = \text{Aut}(\Omega) (= \text{Aut}(H_\Omega(\mathbf{n}; \mathbf{p}))).$$

This means that Corollary 1.2 holds for the unit ball case of Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p})$  in its standard form (cf. Theorem 1.1 in Ahn-Byun-Park [1]).

The second purpose of this paper is to study proper holomorphic mappings between Hua domains. We first recall the structure of proper holomorphic self-mappings of the unit ball  $\mathbf{B}^n$  in  $\mathbb{C}^n$ . When  $n = 1$ , such maps are precisely the finite Blaschke products. The situation is quite different for  $n \geq 2$ . The following fundamental result was proved by Alexander [2] in 1977.

**Theorem 1.C** (Alexander [2]) *Any proper holomorphic self-mapping of the unit ball  $\mathbf{B}^n$  in  $\mathbb{C}^n$  ( $n \geq 2$ ) is an automorphism of  $\mathbf{B}^n$ .*

We remark that

$$f(z_1, z_2) = (z_1, z_2^2) : |z_1|^2 + |z_2|^4 < 1 \longrightarrow |w_1|^2 + |w_2|^2 < 1$$

is a proper holomorphic mapping between two bounded pseudoconvex domains in  $\mathbb{C}^2$  with smooth real-analytic boundary, but it is branched and is not biholomorphic. Thus it suggests a subject to discover some interesting bounded weakly pseudoconvex domains  $D_1, D_2$  in  $\mathbb{C}^n$  ( $n \geq 2$ ) such that any proper holomorphic mapping from  $D_1$  to  $D_2$  is a biholomorphism. There are many important results concerning proper holomorphic mapping  $f : D_1 \rightarrow D_2$  between two bounded pseudoconvex domains  $D_1, D_2$  in  $\mathbb{C}^n$  with smooth boundary. If the proper holomorphic mapping  $f$  extends smoothly to the closure of  $D_1$ , then the extended mapping takes the boundary  $bD_1$  into the boundary  $bD_2$ , and it satisfies the tangential Cauchy-Riemann equations on  $bD_1$ . Thus the proper holomorphic mapping  $f : D_1 \rightarrow D_2$  leads naturally to the geometric study of the mappings from  $bD_1$  into  $bD_2$ . These researches are often heavily based on analytic techniques about the

mapping on boundaries (e.g., see Forstnerič [9] and Huang [13]). The lack of boundary regularity usually presents a serious analytical difficulty.

As we know, in general, a generalized complex ellipsoid is not strongly pseudoconvex and its boundary is not smooth. Also, there are many results (e.g., Dini-Primicerio [6, 7], Hamada [13] and Landucci [16]) concerning proper holomorphic mappings between two generalized complex ellipsoids.

For the case of  $\mathbf{p}, \mathbf{q} \in (\mathbb{Z}_+)^r$ , in 1997, Dini-Primicerio ([7], Th. 4.6) proved the following result.

**Theorem 1.D** (Dini-Primicerio [7]) *Let  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$  be two equidimensional generalized complex ellipsoids with  $\mathbf{n}, \mathbf{m} \in \mathbb{N}^r$  and  $\mathbf{p}, \mathbf{q} \in (\mathbb{Z}_+)^r$  (where  $p_k \neq 1, q_k \neq 1$  for  $2 \leq k \leq r$ ) such that  $n_k \geq 2$  whenever  $p_k \geq 2$  and  $m_k \geq 2$  whenever  $q_k \geq 2$  for  $1 \leq k \leq r$ . Then there exists a proper holomorphic mapping  $f : \Sigma(\mathbf{n}; \mathbf{p}) \rightarrow \Sigma(\mathbf{m}; \mathbf{q})$  if and only if there exists a permutation  $\sigma \in S_r$  such that  $n_{\sigma(j)} = m_j, p_{\sigma(j)} = q_j$  for  $1 \leq j \leq r$ .*

Remark. When  $\mathbf{p}, \mathbf{q} \in (\mathbb{Z}_+)^r$ , we have that  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$  are pseudoconvex domains with real analytic boundaries. Theorem 1.D comes from Theorem 4.6 in Dini-Primicerio [7]. In Dini-Primicerio [7], Theorem 4.6 is proved by Theorem 3.1 (in Dini-Primicerio [7]) assuming that “the sets of weak pseudoconvexity of  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$  are contained in analytic sets of codimension at least 2”, which is equivalent to “ $n_k \geq 2$  whenever  $p_k \geq 2$  and  $m_k \geq 2$  whenever  $q_k \geq 2$  for  $1 \leq k \leq r$ ” in Theorem 1.D (see (2.2) in [7] for references).

Following the methods of Pinchuk [22], Dini-Primicerio [7] proved the so called “localization principle of biholomorphisms” for generalized complex ellipsoids, that is, any local biholomorphism sending boundary points to boundary points extends to a global one, and, as its application, Dini-Primicerio [7] get Theorem 1.D. The approach of Pinchuk [22] to “localization principle of biholomorphisms” is firstly to show that the local biholomorphism is rational (thus extends naturally to be globally meromorphic), and then to show that the rational mapping is biholomorphic by the standard argument: if the zero locus of the holomorphic Jacobian determinant of the rational mapping is nonempty, then the set of points of weak pseudoconvexity should contain a set of real codimension 3. Thus the assumption that “the sets of weak pseudoconvexity is contained in some complex analytic set of complex codimension at least 2” will force the zero locus of the holomorphic Jacobian determinant to be empty. Thus the conditions “ $\mathbf{p}, \mathbf{q} \in (\mathbb{Z}_+)^r$ ” and “ $n_i \geq 2$  whenever  $p_i \geq 2$  and  $m_i \geq 2$  whenever  $q_i \geq 2$  for  $1 \leq i \leq r$ ” are indispensable in proving Theorem 1.D.

Even though the bounded homogeneous domains in  $\mathbb{C}^n$  are always pseudoconvex, there are, of course, many such domains (e.g., all bounded symmetric domains of rank  $\geq 2$ ) such that they do not have smooth boundary and have no strongly pseudoconvex boundary point by the Wong-Rosay theorem (see Rudin [24], Theorem 15.5.10 and its Corollary). There are many rigidity results about the proper holomorphic mappings between bounded symmetric domains.

In 1984, by using results of Bell [4] and Tumanov-Henkin [29], Henkin-Novikov [11] proved the following result (see Th.3.3 in Forstnerič [9] for references).

**Theorem 1.E** (Henkin-Novikov [11]) *Any proper holomorphic self-mapping on an irreducible bounded symmetric domain of rank  $\geq 2$  is an analytic automorphism.*

Using the idea in Mok-Tsai [20] and Tsai [25], Tu [26, 27] (one of the authors of the current article) obtained rigidity results on proper holomorphic mappings between bounded symmetric domains and proved the following in 2002.

**Theorem 1.F** (Tu [26]) *Let  $\Omega_1$  and  $\Omega_2$  be two equidimensional bounded symmetric domains. Assume that  $\Omega_1$  is irreducible and  $\text{rank}(\Omega_1) \geq 2$ . Then, any proper holomorphic mapping from  $\Omega_1$  to  $\Omega_2$  is a biholomorphism.*

Further, using the idea in Mok-Tsai [20] and Tsai [25], in 2010, Mok-Ng-Tu [19] obtained some

rigidity results of proper holomorphic mappings on bounded symmetric domains as follows.

**Theorem 1.G** (Mok-Ng-Tu [19]) *Let  $\Omega$  be an irreducible bounded symmetric domain of rank  $\geq 2$  which is not isomorphic to a Type-IV classical symmetric domain  $D_{IV}^N$  of dimension  $N \geq 3$ . Let  $F : \Omega \rightarrow D$  be a proper holomorphic map onto a bounded convex domain  $D$ . Then,  $F : \Omega \rightarrow D$  is a biholomorphism and  $D$  is, up to an affine-linear transformation, the Harish-Chandra realization of  $\Omega$ .*

The second goal of this paper is to establish what seems to be the first rigidity result for proper holomorphic mappings on Hua domains.

For a Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p}) = H_\Omega(n_1, \dots, n_r; p_1, \dots, p_r)$  in its standard form. The boundary  $bH_\Omega(\mathbf{n}; \mathbf{p})$  of  $H_\Omega(\mathbf{n}; \mathbf{p})$  is comprised of

$$bH_\Omega(\mathbf{n}; \mathbf{p}) = b_0H_\Omega(\mathbf{n}; \mathbf{p}) \cup b_1H_\Omega(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\}), \quad (4)$$

where

$$\begin{aligned} b_0H_\Omega(\mathbf{n}; \mathbf{p}) &:= \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in \Omega \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \right. \\ &\quad \left. \sum_{i=1}^r \|w_{(i)}\|^{2p_i} = N_\Omega(z, z), \|w_{(j)}\|^2 \neq 0, 1 + \delta \leq j \leq r \right\}, \\ b_1H_\Omega(\mathbf{n}; \mathbf{p}) &:= \bigcup_{j=1+\delta}^r \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in \Omega \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \right. \\ &\quad \left. \sum_{i=1}^r \|w_{(i)}\|^{2p_i} = N_\Omega(z, z), \|w_{(j)}\|^2 = 0 \right\}, \end{aligned}$$

in which

$$\delta = \begin{cases} 1 & \text{if } p_1 = 1, \\ 0 & \text{if } p_1 \neq 1. \end{cases}$$

Then we have (by Proposition 2.4 in this paper):

(a)  $b_0H_\Omega(\mathbf{n}; \mathbf{p})$  is a real analytic hypersurface in  $\mathbb{C}^{d+|\mathbf{n}|}$  and  $H_\Omega(\mathbf{n}; \mathbf{p})$  is strongly pseudoconvex at all points of  $b_0H_\Omega(\mathbf{n}; \mathbf{p})$ .

(b) If  $H_\Omega(\mathbf{n}; \mathbf{p})$  isn't a ball, then  $H_\Omega(\mathbf{n}; \mathbf{p})$  is not strongly pseudoconvex at any point of  $b_1H_\Omega(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\})$ .

Obviously,  $b_1H_\Omega(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\})$  is contained in a complex analytic subset in  $\mathbb{C}^{d+|\mathbf{n}|}$  of complex codimension  $\min\{n_{1+\delta}, \dots, n_r, n_1 + \dots + n_r\}$  (note  $\min\{n_{1+\delta}, \dots, n_r, n_1 + \dots + n_r\} = n_1$  for  $r = 1$  and  $\min\{n_{1+\delta}, \dots, n_r, n_1 + \dots + n_r\} = \min\{n_{1+\delta}, \dots, n_r\}$  for  $r \geq 2$ ).

**Theorem 1.3.** *Suppose that*

$$f : H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \rightarrow H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)$$

*is a proper holomorphic mapping between two equidimensional Hua domains  $H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)$  and  $H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)$  in their standard forms, where  $\Omega_1 \subset \mathbb{C}^{d_1}$  and  $\Omega_2 \subset \mathbb{C}^{d_2}$  are two irreducible bounded symmetric domains in the Harish-Chandra realization, and  $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{N}^r$ ,  $\mathbf{p}_1, \mathbf{p}_2 \in (\mathbb{R}_+)^r$ . Assume that  $b_1H_{\Omega_i}(\mathbf{n}_i; \mathbf{p}_i) \cup (b\Omega_i \times \{0\})$  ( $i = 1, 2$ ) is contained in some complex analytic set of complex codimension at least 2. Then  $f : H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \rightarrow H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)$  is a biholomorphism.*

Remarks on Theorem 1.3. (i) In Theorem 1.3, we don't assume  $\dim \Omega_1 = \dim \Omega_2$ .

(ii) In Theorem 1.3, the assumption " $b_1H_\Omega(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\})$  is contained in some complex analytic set of complex codimension at least 2" is equivalent to that  $H_\Omega(\mathbf{n}; \mathbf{p})$  (in its standard form) satisfies

$$\min\{n_{1+\delta}, \dots, n_r, n_1 + \dots + n_r\} \geq 2,$$

that is,  $H_\Omega(\mathbf{n}; \mathbf{p})$  (in its standard form) satisfies the following assumptions: (a) If  $\Omega = \mathbf{B}_d$  is the unit ball, then  $\min\{n_1, \dots, n_r\} \geq 2$ ; (b) If  $\text{rank}(\Omega) \geq 2$  and  $p_1 \neq 1$ , then  $\min\{n_1, \dots, n_r\} \geq 2$ ; (c) If  $\text{rank}(\Omega) \geq 2$  and  $p_1 = 1$ , then  $\min\{n_2, \dots, n_r, n_1 + n_2 + \dots + n_r\} \geq 2$ .

(iii) In Theorem 1.3, the assumption “ $b_1 H_{\Omega_i}(\mathbf{n}_i; \mathbf{p}_i) \cup (b\Omega_i \times \{0\})$  ( $i = 1, 2$ ) is contained in some complex analytic set of complex codimension at least 2” cannot be removed. For example, let  $\Omega$  be an irreducible bounded symmetric domain with  $\text{rank}(\Omega) \geq 2$ ,  $n_1 := 1$  (i.e.,  $w_{(1)} \in \mathbb{C}$ ), and

$$\Phi(z, w_{(1)}, w_{(2)}, \dots, w_{(r)}) := (z, w_{(1)}^2, w_{(2)}, \dots, w_{(r)})$$

for  $(z, w_{(1)}, w_{(2)}, \dots, w_{(r)}) \in H_\Omega(1, n_2, \dots, n_r; p_1, p_2, \dots, p_r)$ . Then  $\Phi$  is a proper holomorphic mapping from  $H_\Omega(1, n_2, \dots, n_r; p_1, p_2, \dots, p_r)$  to  $H_\Omega(1, n_2, \dots, n_r; p_1/2, p_2, \dots, p_r)$ , but  $\Phi$  is not a biholomorphism.

Combining Theorem 1.3 and Corollary 1.2, we immediately have the result as follows.

**Corollary 1.4.** *Suppose that  $f$  is a proper holomorphic self-mapping on the Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p})$  in its standard form, where  $\Omega \subset \mathbb{C}^d$  is an irreducible bounded symmetric domain in the Harish-Chandra realization, and  $\mathbf{n} \in \mathbb{N}^r$ ,  $\mathbf{p} \in (\mathbb{R}_+)^r$  with  $\min\{n_{1+\delta}, \dots, n_r, n_1 + n_2 + \dots + n_r\} \geq 2$ . Then  $f$  is an automorphism of the Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p})$ , that is, there exist a  $\Phi \in \Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$  and a permutation  $\sigma \in S_r$  with  $n_{\sigma(i)} = n_i$ ,  $p_{\sigma(i)} = p_i$  for  $1 \leq i \leq r$  such that*

$$f(z, w_{(1)}, \dots, w_{(r)}) = \Phi(z, w_{(\sigma(1))}, \dots, w_{(\sigma(r))}).$$

When  $\Omega \subset \mathbb{C}^d$  is the unit ball  $\mathbf{B}^d$ , we get that  $H_{\mathbf{B}^d}(\mathbf{n}; \mathbf{p}) = \Sigma((d, \mathbf{n}); (1, \mathbf{p}))$  is a generalized complex ellipsoid. Thus, by Theorem 1.3 and Theorem 1.1, we get the following result about proper holomorphic mappings between generalized complex ellipsoids.

**Corollary 1.5.** *Let  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$  be two equidimensional generalized complex ellipsoids with  $\mathbf{n}, \mathbf{m} \in \mathbb{N}^r$  and  $\mathbf{p}, \mathbf{q} \in (\mathbb{R}_+)^r$  (where  $p_k \neq 1$ ,  $q_k \neq 1$  for  $2 \leq k \leq r$ ). Assume that  $n_i \geq 2$ ,  $m_i \geq 2$  for  $2 \leq i \leq r$  and  $p_1 = 1$ ,  $q_1 = 1$ . Then there exists a proper holomorphic mapping  $f : \Sigma(\mathbf{n}; \mathbf{p}) \rightarrow \Sigma(\mathbf{m}; \mathbf{q})$  if and only if there exists a permutation  $\sigma \in S_r$  such that  $n_{\sigma(j)} = m_j$ ,  $p_{\sigma(j)} = q_j$  for  $1 \leq j \leq r$ .*

Remark. When  $\mathbf{p}, \mathbf{q} \in (\mathbb{R}_+)^r$ , we have that, in general,  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$  are pseudoconvex domains without smooth boundaries. Corollary 1.5 is an extension of Theorem 1.D to the special case of  $\mathbf{p}, \mathbf{q} \in (\mathbb{R}_+)^r$ .

Now we shall present an outline of the argument in our proof of main results.

In general, a Hua domain is a nonhomogeneous domain without smooth boundary. But it is still a bounded complete circular domain. Let

$$f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}, \mathbf{q})$$

be a proper holomorphic mapping between two equidimensional Hua domains in their standard forms. We want to prove that  $f$  is a biholomorphism, and further, to determine the explicit form of the biholomorphism  $f$ .

In order to prove that  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}, \mathbf{q})$  is a biholomorphism, it suffices to show that  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}, \mathbf{q})$  is unbranched. The transformation rule for Bergman kernels under proper holomorphic mapping (e.g., Th. 1 in Bell [5]) plays a key role in extending proper holomorphic mapping. Our idea here is heavily based on the framework of Bell [4, 5] and Pinčuk [22]. The first is to prove that  $f$  extends holomorphically to the closures. By using a kind of semi-regularity at the boundary of the Bergman kernel associated to a Hua domain, we get the extension by using the standard argument in Bell [4]. The second is to prove that  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}, \mathbf{q})$  is unbranched assuming the first one is achieved. By investigating the strongly pseudoconvex part of



the boundary of the Hua domains and using the local regularity for the mappings between strongly pseudoconvex hypersurfaces (e.g., see Pinčuk [22]), we get that  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  is unbranched. So  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  is a biholomorphism. Furthermore, by the uniqueness theorem, we have

$$f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$$

extends to a biholomorphism between their closures.

Next we show that  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  maps the base space to the base space (that is,  $f(\Omega_1 \times \{0\}) \subset \Omega_2 \times \{0\}$ ). Let

$$bH_{\Omega_1}(\mathbf{n}; \mathbf{p}) := b_0H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \cup b_1H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \cup (b\Omega_1 \times \{0\}),$$

$$bH_{\Omega_2}(\mathbf{m}; \mathbf{q}) := b_0H_{\Omega_2}(\mathbf{m}; \mathbf{q}) \cup b_1H_{\Omega_2}(\mathbf{m}; \mathbf{q}) \cup (b\Omega_2 \times \{0\}),$$

where see (4) for the notations. Then (a)  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  (resp.,  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$ ) is strongly pseudoconvex at all points of  $b_0H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  (resp.,  $b_0H_{\Omega_2}(\mathbf{m}; \mathbf{q})$ ); (b) If  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  (resp.,  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$ ) isn't a ball, then  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  (resp.,  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$ ) is not strongly pseudoconvex at any point of  $b_1H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \cup (b\Omega_1 \times \{0\})$  (resp.,  $b_1H_{\Omega_2}(\mathbf{m}; \mathbf{q}) \cup (b\Omega_2 \times \{0\})$ ). By investigating the subset

$$b_1H_{\Omega}(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\})$$

of the boundary  $bH_{\Omega}(\mathbf{n}; \mathbf{p})$  of a Hua domain  $H_{\Omega}(\mathbf{n}; \mathbf{p})$ , we have that  $b_1H_{\Omega}(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\})$  consists of  $r - \delta$  components  $bPr_j(H_{\Omega}(\mathbf{n}; \mathbf{p}))$  for  $1 + \delta \leq j \leq r$  (see (4) for the notation of  $\delta$ ) and  $b\Omega \times \{0\}$  is the intersection of these  $r - \delta$  components, where

$$Pr_j(H_{\Omega}(\mathbf{n}; \mathbf{p})) := H_{\Omega}(\mathbf{n}; \mathbf{p}) \cap \{w_{(j)} = 0\}$$

for  $1 + \delta \leq j \leq r$ . Since  $f$  is a biholomorphism between their closures,  $f$  maps the subset  $b_1H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \cup (b\Omega_1 \times \{0\})$  of  $bH_{\Omega_1}(\mathbf{n}; \mathbf{p})$  onto the subset  $b_1H_{\Omega_2}(\mathbf{m}; \mathbf{q}) \cup (b\Omega_2 \times \{0\})$  of  $bH_{\Omega_2}(\mathbf{m}; \mathbf{q})$ . Apply this fact to  $f$ ,  $Pr_{1+\delta} \circ f$ ,  $Pr_{2+\delta} \circ Pr_{1+\delta} \circ f$ ,  $\dots$ ,  $Pr_r \circ \dots \circ Pr_{1+\delta} \circ f$  in succession, we get

$$f(b\Omega_1 \times \{0\}) \subset b\Omega_2 \times \{0\}.$$

Thus

$$f(\Omega_1 \times \{0\}) \subset \Omega_2 \times \{0\}$$

by the maximum modulus principle. In particular, we have  $f(0, 0) \in \Omega_2 \times \{0\}$ , thus, using fact that  $\Gamma(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))$  is transitive on  $\Omega_2 \times \{0\} (\subset H_{\Omega_2}(\mathbf{m}; \mathbf{q}))$ , we can choose an automorphism  $\Phi \in \Gamma(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))$  (see (3)) with  $\Phi(f(0, 0)) = (0, 0)$ . Thus

$$\Phi \circ f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$$

is a biholomorphism with  $\Phi \circ f(0, 0) = (0, 0)$ , therefore, a holomorphic linear isomorphism by the Cartan's theorem.

At last, we prove that, after a permutation of coordinates, the  $(r + 1) \times (r + 1)$  block matrix of the holomorphic linear isomorphism  $\Phi \circ f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  is a block diagonal matrix. That is, we prove that there exists one and only one nonzero block in every row of the block matrix. Denote the projection by

$$Pr : H_{\Omega_2}(\mathbf{m}; \mathbf{q}) \rightarrow \{0\} \times \Sigma(\mathbf{m}; \mathbf{q}) \quad (:= H_{\Omega_2}(\mathbf{m}; \mathbf{q}) \cap (\{0\} \times \mathbb{C}^{|\mathbf{m}|})).$$

Then we prove that

$$Pr \circ \Phi \circ f \big|_{\overline{\{0\} \times \Sigma(\mathbf{n}; \mathbf{p})}} : \{0\} \times \Sigma(\mathbf{n}; \mathbf{p}) \rightarrow \{0\} \times \Sigma(\mathbf{m}; \mathbf{q})$$

must be a holomorphic linear isomorphism between two generalized complex ellipsoids  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$ , and its matrix  $D$  can be obtained from the block matrix of  $\Phi \circ f$  by deleting the

first row and first column. In order to show that  $D$  is a block diagonal matrix, we argue by the contradiction. If there exist no nonzero block or at least two nonzero blocks  $D_{i_1 j}, D_{i_2 j}$  of some column  $D_j$  of  $D$ , then we have that some strongly pseudoconvex points on  $b\Sigma(\mathbf{n}; \mathbf{p})$  are mapped by  $Pr \circ \Phi \circ f|_{\overline{\{0\} \times \Sigma(\mathbf{n}; \mathbf{q})}}$  to weakly pseudoconvex points on  $b\Sigma(\mathbf{m}; \mathbf{q})$ . This is impossible since it is a holomorphic linear isomorphism. Thus, there exists one and only one nonzero block in every row of the block matrix  $D$ . Further, we prove every block except the first one on the first row and the first column of the matrix of  $\Phi \circ f$  is zero. Thus, we get that after a permutation of coordinates, the  $(r+1) \times (r+1)$  block matrix of the linear isomorphism  $\Phi \circ f$  is a block diagonal matrix. These are the key ideas in proving our main results in this paper.

## 2 Preliminaries

### 2.1 Holomorphic extensions of proper holomorphic mappings

**Proposition 2.1** *Let  $\Omega$  be an irreducible bounded symmetric domain in  $\mathbb{C}^d$  of genus  $g$  in its Harish-Chandra realization and let  $N_\Omega(z, \bar{z})$  be the generic norm of  $\Omega$ . Then we have the results as follows:*

- (a) *For any  $z_0 \in \Omega$ , we have  $N_\Omega(tz_0, \overline{tz_0})$  ( $0 \leq t \leq 1$ ) is a decreasing function of  $t$ .*
- (b) *We have*

$$N_\Omega(z, 0) = 1 \text{ and } 0 < N_\Omega(z, \bar{z}) \leq 1 \text{ } (z \in \Omega),$$

and  $N_\Omega(z, \bar{z}) = 1$  if and only if  $z = 0$ .

- (c) *Let  $H_\Omega(\mathbf{n}; \mathbf{p})$  be a Hua domain. Then, for any  $(z_0, w_{(1)0}, \dots, w_{(r)0}) \in H_\Omega(\mathbf{n}; \mathbf{p})$  and  $0 \leq t \leq 1$ , we have  $(tz_0, tw_{(1)0}, \dots, tw_{(r)0}) \in H_\Omega(\mathbf{n}; \mathbf{p})$ . Therefore, each Hua domain is a starlike domain with respect to the origin of  $\mathbb{C}^{d+|\mathbf{n}|}$ , where  $|\mathbf{n}| := n_1 + \dots + n_r$ .*

**Proof.** Since  $\Omega$  is a bounded circular domain and contains the origin, there is a homogeneous holomorphic polynomial set

$$\left\{ \frac{1}{\sqrt{V(\Omega)}}, h_1(z), h_2(z), \dots \right\},$$

which is an orthonormal basis of the Hilbert space  $A^2(\Omega)$  of square-integrable holomorphic functions on  $\Omega$ , where  $\deg h_j(z) \geq 1$  (so  $h_j(0) = 0$ ) for  $j = 1, 2, \dots$ . Then

$$K_\Omega(z, \bar{\xi}) = \frac{1}{V(\Omega)} + h_1(z)\overline{h_1(\xi)} + h_2(z)\overline{h_2(\xi)} + \dots \quad (5)$$

for all  $z, \xi \in \Omega$ .

- (a) For any  $z_0 \in \Omega$  and  $0 \leq t \leq 1$ , from (5), we have

$$K_\Omega(tz_0, \overline{tz_0}) = \frac{1}{V(\Omega)} + t^{2\deg h_1} |h_1(z_0)|^2 + t^{2\deg h_2} |h_2(z_0)|^2 + \dots$$

is an increasing function of  $t$  ( $0 \leq t \leq 1$ ). So

$$N_\Omega(tz_0, \overline{tz_0}) = (V(\Omega)K_\Omega(tz_0, \overline{tz_0}))^{-1/g}$$

is a decreasing function of  $t$  ( $0 \leq t \leq 1$ ). The proof of Proposition 2.1 (a) is completed.

- (b) Thus, from (5), we have

$$N_\Omega(z, 0) = (V(\Omega)K_\Omega(z, 0))^{-1/g} = 1$$

for all  $z \in \Omega$  and

$$1 \leq V(\Omega)K(z, \bar{z}) < +\infty \text{ } (z \in \Omega) \text{ and } V(\Omega)K(0, 0) = 1.$$

Since  $\Omega \subset \mathbb{C}^d$  is a bounded circular domain and contains the origin, we have that

$$\left\{ \frac{1}{\sqrt{V(\Omega)}}, z_1, (z_2)^2, \dots, (z_d)^d \right\}$$

is an orthogonal set of the Hilbert space  $A^2(\Omega)$  of square-integrable holomorphic functions on  $\Omega$ . Take positive numbers  $r_1, \dots, r_d$  such that

$$\left\{ \frac{1}{\sqrt{V(\Omega)}}, r_1 z_1, r_2 (z_2)^2, \dots, r_d (z_d)^d \right\}$$

is an orthonormal set of  $A^2(\Omega)$ . Then

$$K_\Omega(z, \bar{z}) \geq \frac{1}{V(\Omega)} + r_1^2 |z_1|^2 + r_2^2 |z_2|^4 + \dots + r_d^2 |z_d|^{2d}$$

for all  $z = (z_1, \dots, z_d) \in \Omega$ . Thus  $V(\Omega)K_\Omega(z, \bar{z}) = 1$  ( $z \in \Omega$ ) implies  $z = 0$ .

Therefore, we have  $0 < N_\Omega(z, \bar{z}) \leq 1$  ( $z \in \Omega$ ), and  $N_\Omega(z, \bar{z}) = 1$  if and only if  $z = 0$ . The proof of Proposition 2.1 (b) is completed.

(c) For any  $(z_0, w_{(1)0}, \dots, w_{(r)0}) \in H_\Omega(\mathbf{n}; \mathbf{p})$  and  $0 \leq t \leq 1$ , by definition, we have

$$\|w_{(1)0}\|^{2p_1} + \dots + \|w_{(r)0}\|^{2p_r} < N_\Omega(z_0, \bar{z}_0).$$

Thus, by (a), we have

$$\|tw_{(1)0}\|^{2p_1} + \dots + \|tw_{(r)0}\|^{2p_r} \leq \|w_{(1)0}\|^{2p_1} + \dots + \|w_{(r)0}\|^{2p_r} < N_\Omega(z_0, \bar{z}_0) < N_\Omega(tz_0, \bar{tz}_0).$$

So we get  $(tz_0, tw_{(1)0}, \dots, tw_{(r)0}) \in H_\Omega(\mathbf{n}; \mathbf{p})$ . The proof of Proposition 2.1 (c) is completed.

In order to prove that a proper holomorphic mapping between two equidimensional Hua domains extends holomorphically to their closures, we need the following lemma.

**Lemma 2.2.** (Bell [4], Theorem 2) *Suppose  $f : \Omega_1 \rightarrow \Omega_2$  is a proper holomorphic mapping between bounded circular domains in  $\mathbb{C}^n$ . Suppose further that  $\Omega_2$  contains the origin and that the Bergman kernel function  $K_{\Omega_1}(z, \bar{\xi})$  associated to  $\Omega_1$  is such that for each compact subset  $E$  of  $\Omega_1$ , there is an open set  $U = U(E)$  containing  $\overline{\Omega_1}$  such that  $K_{\Omega_1}(z, \bar{\xi})$  extends to be holomorphic on  $U$  as a function of  $z$  for each  $\xi \in E$ . Then  $f$  extends holomorphically to a neighborhood of  $\overline{\Omega_1}$ .*

Now we prove that a proper holomorphic mapping between two equidimensional Hua domains extends holomorphically to their closures as follows (see Lemma 1.1.1 in Mok [18] and Th. 2.5 in Tu-Wang [28] for references).

**Proposition 2.3.** *Let  $H_\Omega(\mathbf{n}; \mathbf{p}) \subset \mathbb{C}^{d+|\mathbf{n}|}$  be a Hua domain and  $G \subset \mathbb{C}^{d+|\mathbf{n}|}$  be a bounded circular domain containing the origin. Suppose that  $F : H_\Omega(\mathbf{n}; \mathbf{p}) \rightarrow G$  is a proper holomorphic mapping. Then  $F$  extends holomorphically to a neighborhood of  $\overline{H_\Omega(\mathbf{n}; \mathbf{p})}$ .*

**Proof.** Let  $r$  be a real number with  $0 < r < 1$ . Since  $H_\Omega(\mathbf{n}; \mathbf{p}) \subset \mathbb{C}^{d+|\mathbf{n}|}$  is a starlike domain by Proposition 2.1 (c), we have  $rH_\Omega(\mathbf{n}; \mathbf{p}) \subset H_\Omega(\mathbf{n}; \mathbf{p})$ .

Consider the Taylor expansion of the Bergman kernel  $K_{H_\Omega(\mathbf{n}; \mathbf{p})}(z, \bar{\xi})$  on  $H_\Omega(\mathbf{n}; \mathbf{p})$  in  $z = (z_1, \dots, z_{d+|\mathbf{n}|})$  and  $\bar{\xi} = (\bar{\xi}_1, \dots, \bar{\xi}_{d+|\mathbf{n}|})$ . From the invariance of  $H_\Omega(\mathbf{n}; \mathbf{p})$  under the circle group action  $z \rightarrow e^{\sqrt{-1}\theta} z$  ( $\theta \in \mathbb{R}$ ), we have the Bergman kernel  $K_{H_\Omega(\mathbf{n}; \mathbf{p})}(z, \bar{\xi})$  on  $H_\Omega(\mathbf{n}; \mathbf{p})$  is invariant under the circle group action. It follows that the coefficient of  $z^I \bar{\xi}^J$  is zero whenever  $|I| \neq |J|$ . Thus, the Bergman kernel  $K_{H_\Omega(\mathbf{n}; \mathbf{p})}(z, \bar{\xi})$  on  $H_\Omega(\mathbf{n}; \mathbf{p})$  is of the form

$$K_{H_\Omega(\mathbf{n}; \mathbf{p})}(z, \bar{\xi}) = \sum_{|I|=|J|} a_{IJ} z^I \bar{\xi}^J$$

for  $z, \xi \in H_\Omega(\mathbf{n}; \mathbf{p})$ . Since  $(rz)^I \overline{(\xi/r)^J} = z^I \overline{\xi^J}$  whenever  $|I| = |J|$ , we have

$$K_{H_\Omega(\mathbf{n}; \mathbf{p})}(z, \bar{\xi}) = K_{H_\Omega(\mathbf{n}; \mathbf{p})}(rz, \overline{\xi/r})$$

for all  $z \in H_\Omega(\mathbf{n}; \mathbf{p})$ ,  $\xi \in rH_\Omega(\mathbf{n}; \mathbf{p})$ . Then, for every fixed  $\xi \in rH_\Omega(\mathbf{n}; \mathbf{p})$ , we have  $K_{H_\Omega(\mathbf{n}; \mathbf{p})}(z, \bar{\xi})$  extends holomorphically to  $\frac{1}{r}H_\Omega(\mathbf{n}; \mathbf{p})$  as a function of  $z$ .

Therefore, for each compact subset  $E$  of  $H_\Omega(\mathbf{n}; \mathbf{p})$ , there exists a real number  $r_0$  ( $0 < r_0 < 1$ ) with  $E \subset r_0 H_\Omega(\mathbf{n}; \mathbf{p})$  such that  $K_{H_\Omega(\mathbf{n}; \mathbf{p})}(z, \bar{\xi})$  extends holomorphically to  $\frac{1}{r_0}H_\Omega(\mathbf{n}; \mathbf{p})$  (a neighborhood of  $\overline{H_\Omega(\mathbf{n}; \mathbf{p})}$ ) as a function of  $z$  for all  $\xi \in E$ . By Lemma 2.2, we have that  $f$  extends holomorphically to a neighborhood of  $\overline{H_\Omega(\mathbf{n}; \mathbf{p})}$ . The proof of Proposition 2.3 is finished.

## 2.2 The structure of the boundary of a Hua domain $H_\Omega(\mathbf{n}, \mathbf{p})$

For a Hua domain  $H_\Omega(\mathbf{n}; \mathbf{p}) = H_\Omega(n_1, \dots, n_r; p_1, \dots, p_r)$  in its standard form, we will investigate the strongly pseudoconvex part of its boundary  $bH_\Omega(\mathbf{n}; \mathbf{p})$  which is comprised of

$$bH_\Omega(\mathbf{n}; \mathbf{p}) = b_0 H_\Omega(\mathbf{n}; \mathbf{p}) \cup b_1 H_\Omega(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\}),$$

where  $b_0 H_\Omega(\mathbf{n}; \mathbf{p})$  and  $b_1 H_\Omega(\mathbf{n}; \mathbf{p})$  are the same as those in (4).

**Proposition 2.4.** *Let  $\Omega \subset \mathbb{C}^d$  be an irreducible bounded symmetric domain of genus  $g$  in its Harish-Chandra realization. Then we have the conclusions as follows.*

(a)  $b_0 H_\Omega(\mathbf{n}; \mathbf{p})$  is a real analytic hypersurface in  $\mathbb{C}^{d+|\mathbf{n}|}$  and  $H_\Omega(\mathbf{n}; \mathbf{p})$  is strongly pseudoconvex at all points of  $b_0 H_\Omega(\mathbf{n}; \mathbf{p})$ .

(b) If  $H_\Omega(\mathbf{n}; \mathbf{p})$  isn't a ball, then  $H_\Omega(\mathbf{n}; \mathbf{p})$  is not strongly pseudoconvex at any point of

$$b_1 H_\Omega(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\}).$$

Obviously,  $b_1 H_\Omega(\mathbf{n}; \mathbf{p}) \cup (b\Omega \times \{0\})$  is contained in a complex analytic set of complex codimension  $\min\{n_{1+\delta}, \dots, n_r, n_1 + \dots + n_r\}$ . (cf. Lemma 3.2 in Rong [23].)

**Proof.** Let  $\{h_i(z)\}_{i=1}^\infty$  be an orthonormal basis of the Hilbert space  $A^2(\Omega)$  of square-integrable holomorphic functions. Then we have

$$K_\Omega(z, \bar{z}) = \sum_{i=1}^\infty h_i(z) \overline{h_i(z)}$$

converges uniformly on any compact subset of  $\Omega$ . Let

$$\rho(z, w_{(1)}, \dots, w_{(r)}) := \|w_{(1)}\|^{2p_1} + \dots + \|w_{(r)}\|^{2p_r} - \sigma(K_\Omega(z, \bar{z}))^{-\lambda}$$

where  $\sigma := (V(\Omega))^{-\frac{1}{g}}$  and  $\lambda := \frac{1}{g}$  are positive. Then  $\rho$  is a real analytic definition function of  $b_0 H_\Omega(\mathbf{n}; \mathbf{p})$ .

Fix a point  $(z_0, w_{(1)0}, \dots, w_{(r)0}) \in b_0 H_\Omega(\mathbf{n}; \mathbf{p}) \subset \mathbb{C}^d \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r}$  and let

$$T = (\xi, \eta_1, \dots, \eta_r) \in T_{(z_0, w_{(1)0}, \dots, w_{(r)0})}^{1,0}(b_0 H_\Omega(\mathbf{n}; \mathbf{p})) \subset \mathbb{C}^d \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r}.$$

Then by definition, we have

$$w_{(j)0} \neq 0, \quad j = 1, \dots, r; \tag{6}$$

$$\|w_{(1)0}\|^{2p_1} + \dots + \|w_{(r)0}\|^{2p_r} - \sigma(K_\Omega(z_0, \bar{z}_0))^{-\lambda} = 0; \tag{7}$$

$$\sum_{k=1}^r p_k \|w_{(k)0}\|^{2(p_k-1)} (\overline{w_{(k)0}} \cdot \eta_k) + \sigma \lambda (K_\Omega(z_0, \bar{z}_0))^{-\lambda-1} \sum_{i=1}^\infty \overline{h_i(z_0)} (h'_i(z_0) \cdot \xi) = 0, \tag{8}$$

where  $h'_i(z_0) \cdot \xi = \sum_{k=1}^d \frac{\partial h_i}{\partial z_k}(z_0) \xi_k$ .

Therefore, from (6),(7),(8), the Levi form of  $\rho$  at the point  $(z_0, w_{(1)0}, \dots, w_{(r)0})$  is computed as follows:

$$\begin{aligned}
L_\rho(T, T) &:= \sum_{i,j=1}^{d+|\mathbf{n}|} \frac{\partial^2 \rho}{\partial T_i \partial \overline{T_j}}(z_0, w_{(1)0}, \dots, w_{(r)0}) T_i \overline{T_j} \\
&= \sum_{k=1}^r p_k \|w_{(k)0}\|^{2(p_k-1)} \|\eta_k\|^2 + \sum_{k=1}^r p_k(p_k-1) \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_k|^2 \\
&\quad + \sigma \lambda K_\Omega(z_0, \overline{z_0})^{-(\lambda+2)} [K_\Omega(z_0, \overline{z_0}) \sum_{i=1}^{\infty} |h'_i(z_0) \cdot \xi|^2 - (\lambda+1) |\sum_{i=1}^{\infty} \overline{h_i(z_0)} (h'_i(z_0) \cdot \xi)|^2] \\
&= \sum_{k=1}^r p_k (\|w_{(k)0}\|^{2(p_k-1)} \|\eta_k\|^2 - \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_k|^2) \\
&\quad + \sum_{k=1}^r p_k^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_k|^2 - \sigma \lambda^2 K_\Omega(z_0, \overline{z_0})^{-(\lambda+2)} |\sum_{i=1}^{\infty} \overline{h_i(z_0)} (h'_i(z_0) \cdot \xi)|^2 \\
&\quad + \sigma \lambda K_\Omega(z_0, \overline{z_0})^{-(\lambda+2)} \left[ K_\Omega(z_0, \overline{z_0}) \sum_{i=1}^{\infty} |h'_i(z_0) \cdot \xi|^2 - |\sum_{i=1}^{\infty} \overline{h_i(z_0)} (h'_i(z_0) \cdot \xi)|^2 \right] \\
&= \sum_{k=1}^r p_k \|w_{(k)0}\|^{2(p_k-2)} \left[ \|w_{(k)0}\|^2 \|\eta_k\|^2 - |\overline{w_{(k)0}} \cdot \eta_k|^2 \right] + \left( \sum_{k=1}^r \|w_{(k)0}\|^2 \right)^{-1} \\
&\quad \times \left[ \left( \sum_{k=1}^r p_k^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_k|^2 \right) \left( \sum_{k=1}^r \|w_{(k)0}\|^2 \right) - \left| \sum_{k=1}^r p_k \|w_{(k)0}\|^{2(p_k-1)} (\overline{w_{(k)0}} \cdot \eta_k) \right|^2 \right] \\
&\quad + \sigma \lambda K_\Omega(z_0, \overline{z_0})^{-(\lambda+2)} \left[ \left( \sum_{i=1}^{\infty} |h_i(z_0)|^2 \right) \left( \sum_{i=1}^{\infty} |h'_i(z_0) \cdot \xi|^2 \right) - \left| \sum_{i=1}^{\infty} \overline{h_i(z_0)} (h'_i(z_0) \cdot \xi) \right|^2 \right] \\
&\geq 0
\end{aligned}$$

by the Cauchy-Schwarz inequality, for all  $T = (\xi, \eta_1, \dots, \eta_r) \in T_{(z_0, w_{(1)0}, \dots, w_{(r)0})}^{1,0}(b_0 H_\Omega(\mathbf{n}; \mathbf{p}))$  and the equality holds if and only if

$$\|w_{(k)0}\|^2 \|\eta_k\|^2 - |\overline{w_{(k)0}} \cdot \eta_k|^2 = 0, \quad 1 \leq k \leq r; \quad (9)$$

$$\left( \sum_{k=1}^r p_k^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_k|^2 \right) \left( \sum_{k=1}^r \|w_{(k)0}\|^2 \right) - \left| \sum_{k=1}^r p_k \|w_{(k)0}\|^{2(p_k-1)} (\overline{w_{(k)0}} \cdot \eta_k) \right|^2 = 0; \quad (10)$$

$$\left( \sum_{i=1}^{\infty} |h_i(z_0)|^2 \right) \left( \sum_{i=1}^{\infty} |h'_i(z_0) \cdot \xi|^2 \right) - \left| \sum_{i=1}^{\infty} \overline{h_i(z_0)} (h'_i(z_0) \cdot \xi) \right|^2 = 0. \quad (11)$$

Now we prove the Levi form  $L_\rho(T, T)$  of  $\rho$  at the point  $(z_0, w_{(1)0}, \dots, w_{(r)0})$  is positive for any nonzero  $T = (\xi, \eta_1, \dots, \eta_r) \in T_{(z_0, w_{(1)0}, \dots, w_{(r)0})}^{1,0}(b_0 H_\Omega(\mathbf{n}; \mathbf{p}))$  as follows:

**Case 1** Suppose  $\xi \neq 0$ . Since

$$K_\Omega(z, \bar{z}) = \sum_{i=1}^{\infty} h_i(z) \overline{h_i(z)}$$

is independent of the choice of the orthonormal basis  $\{h_i(z)\}_{i=1}^{\infty}$  of the Hilbert space  $A^2(\Omega)$  and  $\Omega$  is bounded, we may choose that  $h_1(z)$  is a nonzero constant and  $h_2(z)$  satisfies  $h'_2(z_0) \cdot \xi \neq 0$ . This gives that  $(h_1(z_0), h_2(z_0))$  and  $(h'_1(z_0) \cdot \xi, h'_2(z_0) \cdot \xi)$  are linearly independent. Thus

$$\sum_{k=1}^{\infty} |h_k(z_0)|^2 \sum_{i=1}^{\infty} |h'_i(z_0) \cdot \xi|^2 - \left| \sum_{i=1}^{\infty} \overline{h_i(z_0)} (h'_i(z_0) \cdot \xi) \right|^2 > 0$$

which is a contradiction with (11). Therefore,  $L_\rho(T, T) > 0$  for all

$$T = (\xi, \eta_1, \dots, \eta_r) \in T_{(z_0, w_{(1)0}, \dots, w_{(r)0})}^{1,0}(b_0 H_\Omega(\mathbf{n}; \mathbf{p}))$$

with  $\xi \neq 0$ .

**Case 2** Suppose  $\xi = 0$ . Then  $T = (\xi, \eta_1, \dots, \eta_r) \neq 0$  implies that there exists  $\eta_{i_0} \neq 0$ . On the other hand, since  $\xi = 0$ , by (8), we have

$$\sum_{k=1}^r p_k \|w_{(k)0}\|^{2(p_k-1)} (\overline{w_{(k)0}} \cdot \eta_k) = 0.$$

Hence, by (10), we get

$$\left( \sum_{k=1}^r p_k^2 \|w_{(k)0}\|^{2(p_k-1)} \|\eta_k\|^2 \right) \left( \sum_{k=1}^r \|w_{(k)0}\|^2 \right) = 0.$$

Since  $\|w_{(j)0}\|^2 \neq 0, 1 \leq j \leq r$ , we have  $\eta_i = 0, 1 \leq i \leq r$ , this is a contradiction. Therefore,  $L_\rho(T, T) > 0$  for all

$$T = (0, \eta_1, \dots, \eta_r) \in T_{(z_0, w_{(1)0}, \dots, w_{(r)0})}^{1,0}(b_0 H_\Omega(\mathbf{n}; \mathbf{p}))$$

with  $(\eta_1, \dots, \eta_r) \neq (0, \dots, 0)$ .

Thus the Levi form  $L_\rho(T, T)$  is positive definite on  $T_{(z_0, w_{(1)0}, \dots, w_{(r)0})}^{1,0}(b_0 H_\Omega(\mathbf{n}; \mathbf{p}))$ . This means that every point of  $b_0 H_\Omega(\mathbf{n}; \mathbf{p})$  is strongly pseudoconvex.

Let  $(z_0, w_{(1)0}, \dots, w_{(r)0}) \in b_1 H_\Omega(\mathbf{n}; \mathbf{p})$ . Without loss of generality, we assume  $\|w_{(k)0}\| = 0$  for  $1 + \delta \leq k \leq i_0$  and  $\|w_{(k)0}\| \neq 0$  for  $i_0 + 1 \leq k \leq r$ . Suppose  $p_k \geq 2$  for  $1 + \delta \leq k \leq i_0$  (otherwise,  $bH_\Omega(\mathbf{n}; \mathbf{p})$  is not  $C^2$  at the point  $(z_0, w_{(1)0}, \dots, w_{(r)0})$ , so  $bH_\Omega(\mathbf{n}; \mathbf{p})$  is not strongly pseudoconvex at the point). In this case, take  $T_0 = (0, 0, \dots, \eta_{i_0}, \dots, 0) \in T_{(z_0, w_{(1)0}, \dots, w_{(r)0})}^{1,0}(bH_\Omega(\mathbf{n}; \mathbf{p}))$  with  $\|\eta_{i_0}\| \neq 0$ , then  $L_\rho(T_0, T_0) = 0$ . Hence,  $H_\Omega(\mathbf{n}; \mathbf{p})$  is not strongly pseudoconvex at any point of  $b_1 H_\Omega(\mathbf{n}; \mathbf{p})$ .

For any irreducible bounded symmetric domain  $\Omega$  in  $\mathbb{C}^d$ , we have  $\Gamma(H_\Omega(\mathbf{n}; \mathbf{p}))$  is transitive on  $\Omega \times \{0\} (\subset H_\Omega(\mathbf{n}; \mathbf{p}))$ . Since  $H_\Omega(\mathbf{n}; \mathbf{p})$  is not the unit ball,  $H_\Omega(\mathbf{n}; \mathbf{p})$  is not strictly pseudoconvex at any point of  $b\Omega \times \{0\}$  by the Wong-Rosay theorem. The proof of Proposition 2.4 is completed.

**Lemma 2.5.** (Pinchuk [22], Lemma 1.3) *Let  $D_1, D_2 \subset \mathbb{C}^n$  be two domains,  $p \in bD_1$ , and let  $U$  be a neighborhood of  $p$  in  $\mathbb{C}^n$  such that  $U \cap \overline{D_1}$  is connected. Suppose that the mapping  $f = (f_1, \dots, f_n) : U \cap \overline{D_1} \rightarrow \mathbb{C}^n$  is continuously differentiable in  $U \cap \overline{D_1}$  and holomorphic in  $U \cap D_1$  with  $f(U \cap bD_1) \subset bD_2$ . Take a domain  $V \subset \mathbb{C}^n$  with  $f(U \cap D_1) \subset V$ . Suppose that  $U \cap bD_1$  and  $U \cap bD_2$  are strongly pseudoconvex hypersurfaces in  $\mathbb{C}^n$ . Then either  $f$  is constant or the holomorphic Jacobian determinant  $J_f(z) = \det(\frac{\partial f_i}{\partial z_j})$  does not vanish in  $U \cap bD_1$ .*

**Lemma 2.6.** *Let Hua domains  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  and  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  be in their standard forms, where  $\Omega_1 \subset \mathbb{C}^{d_1}$  and  $\Omega_2 \subset \mathbb{C}^{d_2}$  are two irreducible bounded symmetric domains in the Harish-Chandra realization, and  $\mathbf{n}, \mathbf{m} \in \mathbb{N}^r$ ,  $\mathbf{p}, \mathbf{q} \in (\mathbb{R}_+)^r$ . Then every biholomorphism  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  sends  $\Omega_1 \times \{0\}$  into  $\Omega_2 \times \{0\}$ . Therefore, we have  $f(\Omega_1 \times \{0\}) = \Omega_2 \times \{0\}$ .*

**Proof.** We will divide our proof into the following two cases.

Case 1. Suppose that  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  is a unit ball. Since there exists a biholomorphism  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$ , we have  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  must be the unit ball also.

In fact, since there exists a biholomorphism  $f$  from the unit ball onto  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$ , we have  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  must be a bounded symmetric domain with rank 1. Then  $b\Omega_2 \times \{0\} (\subset bH_{\Omega_2}(\mathbf{m}; \mathbf{q}))$  can not contain any positive-dimensional complex submanifold, and so  $\Omega_2$  is a bounded symmetric domain with rank 1 in the Harish-Chandra realization. This means  $\Omega_2$  is a unit ball. So we have

$$H_{\Omega_2}(\mathbf{m}; \mathbf{q}) = \left\{ (z, w_{(1)}, \dots, w_{(r)}) \in B^d \times \mathbb{C}^{m_1} \times \dots \times \mathbb{C}^{m_r} : \|z\|^2 + \sum_{j=1}^r \|w_{(j)}\|^{2q_j} < 1 \right\}.$$

It is known that the generalized complex ellipsoid is homogeneous if and only if  $q_j = 1$  for all  $1 \leq j \leq r$  (cf. Kodama [14]). Thus we have  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  must be the unit ball.

From Hua domains  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  and  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  being in their standard forms, we get  $H_{\Omega_1}(\mathbf{n}; \mathbf{p}) = \Omega_1 (\cong \Omega_1 \times \{0\})$  and  $H_{\Omega_2}(\mathbf{m}; \mathbf{q}) = \Omega_2 (\cong \Omega_2 \times \{0\})$ . Then, Lemma 2.6 is true.

Case 2. Suppose that  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  is not a unit ball. Let  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  be a biholomorphism. By Proposition 2.3, the biholomorphism

$$f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$$

extends a biholomorphism between  $\overline{H_{\Omega_1}(\mathbf{n}; \mathbf{p})}$  and  $\overline{H_{\Omega_2}(\mathbf{m}; \mathbf{q})}$  by the uniqueness theorem. So, by Proposition 2.4(b), we have

$$f((b\Omega_1 \times \{0\}) \cup b_1 H_{\Omega_1}(\mathbf{n}; \mathbf{p})) = (b\Omega_2 \times \{0\}) \cup b_1 H_{\Omega_2}(\mathbf{m}; \mathbf{q}).$$

Since

$$(b\Omega_1 \times \{0\}) \cup b_1 H_{\Omega_1}(\mathbf{n}; \mathbf{p}) = \bigcup_{j=1+\delta}^r bPr_j(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))$$

and

$$(b\Omega_2 \times \{0\}) \cup b_1 H_{\Omega_2}(\mathbf{m}; \mathbf{q}) = \bigcup_{j=1+\varepsilon}^r bPr_j(H_{\Omega_2}(\mathbf{m}; \mathbf{q})),$$

where

$$Pr_j(H_{\Omega_1}(\mathbf{n}; \mathbf{p})) := H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \cap \{w_{(j)} = 0\} \quad (1 + \delta \leq j \leq r)$$

and

$$Pr_j(H_{\Omega_2}(\mathbf{m}; \mathbf{q})) := H_{\Omega_2}(\mathbf{m}; \mathbf{q}) \cap \{w'_{(j)} = 0\} \quad (1 + \varepsilon \leq j \leq r),$$

in which

$$\delta = \begin{cases} 1 & \text{if } p_1 = 1, \\ 0 & \text{if } p_1 \neq 1, \end{cases} \quad \varepsilon = \begin{cases} 1 & \text{if } q_1 = 1, \\ 0 & \text{if } q_1 \neq 1, \end{cases}$$

we have

$$f\left(\bigcup_{j=1+\delta}^r bPr_j(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))\right) = \bigcup_{j=1+\varepsilon}^r bPr_j(H_{\Omega_2}(\mathbf{m}; \mathbf{q})).$$

In particular, we have

$$f(bPr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))) \subset \bigcup_{j=1+\varepsilon}^r bPr_j(H_{\Omega_2}(\mathbf{m}; \mathbf{q})).$$

Let  $f = (\tilde{f}, f_{(1)}, \dots, f_{(r)})$  and  $U_j := \{\xi \in \overline{Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))} : f_{(j)}(\xi) = 0\}$ . Then

$$bPr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p})) \subset \bigcup_{j=1+\varepsilon}^r U_j.$$

Since  $bPr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))$  has real codimension 1, there exists a  $U_{j_0}$  with real codimension  $\leq 1$ . On the other hand, if  $U_{j_0}$  is a proper complex analytic subset of  $\overline{Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))}$ , then  $U_{j_0}$  has real codimension  $\geq 2$ . Thus there exists a  $U_{j_0}$  such that

$$U_{j_0} = \overline{Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))}.$$

That is,  $f_{(j_0)} \equiv 0$  on  $Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))$ . Hence,

$$f|_{Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))} : Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p})) \rightarrow Pr_{j_0}(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))$$

is a proper holomorphic mapping and holomorphic on the closure of  $Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))$ . If  $n_{1+\delta} < m_{j_0}$ , then  $\dim Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p})) > \dim Pr_{j_0}(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))$ . Since  $f|_{Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))}$  is proper, it is a contradiction. Thus,  $n_{1+\delta} \geq m_{j_0}$ .

Since  $f$  is a biholomorphism, by the similar argument, we have  $f^{-1}(Pr_{j_0}(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))) \subset Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))$  and

$$f^{-1} \mid_{Pr_{j_0}(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))}: Pr_{j_0}(H_{\Omega_2}(\mathbf{m}; \mathbf{q})) \rightarrow Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))$$

is a proper holomorphic mapping and holomorphic on the closure of  $Pr_{j_0}(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))$ . Therefore, we have  $m_{j_0} \geq n_{1+\delta}$  and so we have  $n_{1+\delta} = m_{j_0}$ .

This means that

$$f \mid_{Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))}: Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p})) \rightarrow Pr_{j_0}(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))$$

is a biholomorphism and holomorphic on the closure of  $Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))$ . Thus, we have

$$f \mid_{\overline{Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))}} ((b\Omega_1 \times \{0\}) \cup b_1 Pr_{1+\delta}(H_{\Omega_1}(\mathbf{m}, \mathbf{q}))) \subset (b\Omega_2 \times \{0\}) \cup b_1 Pr_{j_0}(H_{\Omega_2}(\mathbf{m}, \mathbf{q}))$$

by Proposition 2.4(b). Therefore,

$$f \mid_{\overline{Pr_{1+\delta}(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))}} \left( \bigcup_{j=2+\delta}^r bPr_j(Pr_1(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))) \right) \subset \bigcup_{j=1+\varepsilon, j \neq j_0}^r bPr_j(Pr_{j_0}(H_{\Omega_2}(\mathbf{m}; \mathbf{q}))),$$

where

$$Pr_j(Pr_1(H_{\Omega_1}(\mathbf{n}; \mathbf{p}))) = Pr_1(H_{\Omega_1}(\mathbf{n}; \mathbf{p})) \cap \{w_{(j)} = 0\} \quad (2 + \delta \leq j \leq r).$$

By induction, since  $b\Omega_1 \times \{0\} = Pr_r(\cdots Pr_{2+\delta}(Pr_{1+\delta}H_{\Omega_1}(\mathbf{n}; \mathbf{p})))$ , we have  $f \mid_{\overline{\Omega_1 \times \{0\}}} (b\Omega_1 \times \{0\}) \subset (b\Omega_2 \times \{0\})$  and thus

$$f(\Omega_1 \times \{0\}) \subset \Omega_2 \times \{0\}$$

by the maximum modulus principle. The proof of Lemma 2.6 is completed.

Remark. It is important that the Hua domain is written *in its standard form* in Lemma 2.6. For example, define

$$H_{\mathbf{B}^2}((2, 2); (1, 2)) = \{(z, w_{(1)}, w_{(2)}) \in \mathbf{B}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 : \|z\|^2 + \|w_{(1)}\|^2 + \|w_{(2)}\|^4 < 1\}.$$

Then  $H_{\mathbf{B}^2}((2, 2); (1, 2)) = H_{\mathbf{B}^4}((2); (2))$ , and an automorphism

$$\varphi \in \text{Aut}(H_{\mathbf{B}^2}((2, 2); (1, 2))) (= \text{Aut}(H_{\mathbf{B}^4}((2); (2))))$$

sends  $\mathbf{B}^4 \times \{0\}$  into  $\mathbf{B}^4 \times \{0\}$ , but in general, can not send  $\mathbf{B}^2 \times \{0\}$  into  $\mathbf{B}^2 \times \{0\}$  (see Theorem 1.B for references).

### 2.3 Complex linear isomorphisms between two equidimensional generalized complex ellipsoids

In order to get the explicit form of the biholomorphisms between two equidimensional Hua domains, we need following two lemmas about generalized complex ellipsoids.

**Lemma 2.7.** *Let  $D_j \in M_{n_j \times m}$ , ( $1 \leq j \leq r$ ) be  $n_j \times m$  matrix with  $m \leq n_j$  such that the  $(n_1 + \cdots + n_r) \times m$  matrix  $(D_1, \cdots, D_r)^t$  has rank  $m$ . If the system of linear equations*

$$\sum_{j=1}^r \zeta_j D_j = \mathbf{0}, \tag{12}$$

where  $\zeta_j \in \mathbb{C}^{n_j}$  ( $1 \leq j \leq r$ ), does not have solution  $(\alpha_1, \cdots, \alpha_r) \in (\mathbb{C}^{n_1} \setminus \{\mathbf{0}\}) \times \cdots \times (\mathbb{C}^{n_r} \setminus \{\mathbf{0}\})$ , then there exists at least one  $n_j$  such that  $n_j = m$  and only one  $D_{j_0}$  with  $D_{j_0} \neq \mathbf{0}$ . Moreover,  $n_{j_0} = m$  and  $D_{j_0}$  is a nonsingular  $m \times m$  matrix.



**Proof.** Suppose that each  $n_j > m, j = 1, \dots, r$ , then each system of linear equations

$$\zeta_j D_j = \mathbf{0}$$

has a solution  $\alpha_j \in \mathbb{C}^{n_j} \setminus \{\mathbf{0}\}$ . Thus  $(\alpha_1, \dots, \alpha_r) \in (\mathbb{C}^{n_1} \setminus \{\mathbf{0}\}) \times \dots \times (\mathbb{C}^{n_r} \setminus \{\mathbf{0}\})$  is a solution of the system of linear equations (12), a contradiction. Hence there exists at least one  $n_j$  such that  $n_j = m$ .

If all  $D_j$ 's are singular  $m \times m$  matrices, then by the same reasoning as above, we can get a solution  $(\alpha_1, \dots, \alpha_r) \in (\mathbb{C}^{n_1} \setminus \{\mathbf{0}\}) \times \dots \times (\mathbb{C}^{n_r} \setminus \{\mathbf{0}\})$  of the system of linear equations (12), a contradiction. Thus there exists a nonsingular  $m \times m$  matrix, say  $D_1$ .

If there exists another  $D_j$  with  $D_j \neq \mathbf{0}$ , then we can choose  $(\alpha_2, \dots, \alpha_r) \in (\mathbb{C}^{n_2} \setminus \{\mathbf{0}\}) \times \dots \times (\mathbb{C}^{n_r} \setminus \{\mathbf{0}\})$ , such that

$$\sum_{j=2}^r \alpha_j D_j \neq \mathbf{0}.$$

Consider the system of linear equations

$$\zeta_1 D_1 = \sum_{j=2}^r \alpha_j D_j.$$

Since  $D_1$  is nonsingular and  $\sum_{j=2}^r \alpha_j D_j \neq \mathbf{0}$ , it has a unique solution  $\alpha_1 \in \mathbb{C}^{n_1} \setminus \{\mathbf{0}\}$ . Thus  $(\alpha_1, \dots, \alpha_r) \in (\mathbb{C}^{n_1} \setminus \{\mathbf{0}\}) \times \dots \times (\mathbb{C}^{n_r} \setminus \{\mathbf{0}\})$  is a solution of the system of linear equations (12), a contradiction. Thus,  $D_1$  is the unique nonzero matrix. The proof of Lemma 2.7 is finished.

**Lemma 2.8.** *Let  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$  be two equidimensional generalized complex ellipsoids, where  $\mathbf{n}, \mathbf{m} \in \mathbb{N}^r$ ,  $\mathbf{p}, \mathbf{q} \in (\mathbb{R}_+)^r$  (where  $p_k \neq 1, q_k \neq 1$  for  $2 \leq k \leq r$ ). Let  $h : \Sigma(\mathbf{n}; \mathbf{p}) \rightarrow \Sigma(\mathbf{m}; \mathbf{q})$  be a biholomorphic linear isomorphism between  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$ . Then there exists a permutation  $\sigma \in S_r$  such that  $n_{\sigma(i)} = m_i, p_{\sigma(i)} = q_i$  and*

$$h(\zeta_1, \zeta_2, \dots, \zeta_r) = (\zeta_{\sigma(1)}, \zeta_{\sigma(2)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_r \end{pmatrix},$$

where  $U_i$  is a unitary transformation of  $\mathbb{C}^{m_i}$  ( $m_i = n_{\sigma(i)}$ ) for  $1 \leq i \leq r$ .

**Proof.** Let

$$\delta = \begin{cases} 1 & \text{if } p_1 = 1 \\ 0 & \text{if } p_1 \neq 1 \end{cases}, \quad \varepsilon = \begin{cases} 1 & \text{if } q_1 = 1 \\ 0 & \text{if } q_1 \neq 1 \end{cases}.$$

Moreover, we assume that  $n_{1+\delta} \leq \dots \leq n_r$  and  $m_{1+\varepsilon} \leq \dots \leq m_r$ .

Define  $b_0 \Sigma(\mathbf{n}; \mathbf{p})$ ,  $b_1 \Sigma(\mathbf{n}; \mathbf{p})$  and  $b_0 \Sigma(\mathbf{m}; \mathbf{q})$ ,  $b_1 \Sigma(\mathbf{m}; \mathbf{q})$  as following:

$$\begin{aligned} b_0 \Sigma(\mathbf{n}; \mathbf{p}) &:= \left\{ (\zeta_1, \dots, \zeta_r) \in \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \sum_{i=1}^r \|\zeta_i\|^{2p_i} = 1, \|\zeta_j\| \neq 0, 1+\delta \leq j \leq r \right\}, \\ b_1 \Sigma(\mathbf{n}; \mathbf{p}) &:= \bigcup_{j=1+\delta}^r \left\{ (\zeta_1, \dots, \zeta_r) \in \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r} : \sum_{i=1}^r \|\zeta_i\|^{2p_i} = 1, \|\zeta_j\| = 0 \right\}, \\ b_0 \Sigma(\mathbf{m}; \mathbf{q}) &:= \left\{ (\xi_1, \dots, \xi_r) \in \mathbb{C}^{m_1} \times \dots \times \mathbb{C}^{m_r} : \sum_{i=1}^r \|\xi_i\|^{2q_i} = 1, \|\xi_j\| \neq 0, 1+\varepsilon \leq j \leq r \right\}, \\ b_1 \Sigma(\mathbf{m}; \mathbf{q}) &:= \bigcup_{j=1+\varepsilon}^r \left\{ (\xi_1, \dots, \xi_r) \in \mathbb{C}^{m_1} \times \dots \times \mathbb{C}^{m_r} : \sum_{i=1}^r \|\xi_i\|^{2q_i} = 1, \|\xi_j\| = 0 \right\}. \end{aligned}$$

Then  $b_0\Sigma(\mathbf{n}; \mathbf{p})$  (resp.  $b_0\Sigma(\mathbf{m}; \mathbf{q})$ ) consists of all strongly pseudoconvex points of  $b\Sigma(\mathbf{n}; \mathbf{p})$  (resp.  $b\Sigma(\mathbf{m}; \mathbf{q})$ ) and any one of  $b_1\Sigma(\mathbf{n}; \mathbf{p})$  (resp.  $b_1\Sigma(\mathbf{m}; \mathbf{q})$ ) is not a strongly pseudoconvex point of  $b\Sigma(\mathbf{n}; \mathbf{p})$  (resp.  $b\Sigma(\mathbf{m}; \mathbf{q})$ ). Since  $h$  is a biholomorphic linear isomorphism, we have

$$h(b_0\Sigma(\mathbf{n}; \mathbf{p})) = b_0\Sigma(\mathbf{m}; \mathbf{q}) \quad (13)$$

and

$$h(b_1\Sigma(\mathbf{n}; \mathbf{p})) = b_1\Sigma(\mathbf{m}; \mathbf{q}). \quad (14)$$

Let

$$h(\zeta_1, \zeta_2, \dots, \zeta_r) = (\zeta_1, \zeta_2, \dots, \zeta_r) \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1r} \\ D_{21} & D_{22} & \cdots & D_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ D_{r1} & D_{r2} & \cdots & D_{rr} \end{pmatrix}. \quad (15)$$

According to whether  $p_1$  or  $q_1$  is equal to 1 or not, there are two cases: **(i)** Neither  $p_1$  nor  $q_1$  equals to 1; **(ii)** Either  $p_1$  or  $q_1$  equals to 1.

**Case (i).** In this case, we have  $p_1 \neq 1$ ,  $q_1 \neq 1$  and thus  $\delta = 0$ ,  $\varepsilon = 0$ .

$$b_0\Sigma(\mathbf{n}; \mathbf{p}) := \left\{ (\zeta_1, \dots, \zeta_r) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} : \sum_{i=1}^r \|\zeta_i\|^{2p_i} = 1, \|\zeta_j\| \neq 0, 1 \leq j \leq r \right\},$$

$$b_0\Sigma(\mathbf{m}; \mathbf{q}) := \left\{ (\xi_1, \dots, \xi_r) \in \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_r} : \sum_{i=1}^r \|\xi_i\|^{2q_i} = 1, \|\xi_j\| \neq 0, 1 \leq j \leq r \right\}.$$

Since  $h$  a biholomorphic linear isomorphism, we can assume  $m_1 \leq n_1$ . Hence, we have  $m_1 \leq n_1 \leq \cdots \leq n_r$ . In the following we will use Lemma 2.7 to prove that there exist exactly one nonzero block in the first column of the matrix of  $h$ . Consider the system of linear equations

$$\sum_{j=1}^r \zeta_j D_{j1} = 0. \quad (16)$$

Suppose that  $(\alpha_1, \dots, \alpha_r)$  is a solution of (16) with  $\|\alpha_1\|^2 \neq 0, \dots, \|\alpha_r\|^2 \neq 0$ . Then there exists a  $\lambda > 0$  such that

$$\lambda \sum_{j=1}^r \|\alpha_j\|^{2p_j} = 1,$$

that is,  $(\lambda^{\frac{1}{2p_1}} \alpha_1, \dots, \lambda^{\frac{1}{2p_r}} \alpha_r) \in b_0\Sigma(\mathbf{n}; \mathbf{p})$ . But the first component of  $h(\lambda^{\frac{1}{2p_1}} \alpha_1, \dots, \lambda^{\frac{1}{2p_r}} \alpha_r)$  is  $\mathbf{0} \in \mathbb{C}^{m_1}$ , and thus  $h(\lambda^{\frac{1}{2p_1}} \alpha_1, \dots, \lambda^{\frac{1}{2p_r}} \alpha_r) \notin b_0\Sigma(\mathbf{m}; \mathbf{q})$ . This is a contradiction with (13). Thus, the system of linear equations (16) does not have solution  $(\alpha_1, \dots, \alpha_r)$  with  $\|\alpha_1\|^2 \neq 0, \dots, \|\alpha_r\|^2 \neq 0$ . By Lemma 2.7, there is exactly one  $D_{j_1 1} \neq \mathbf{0}$ . After a permutation  $\sigma_1$  of row index of  $D_{ij}$  (which is equivalent to a permutation  $\sigma_1$  of the index of  $\zeta_i$  ( $1 \leq i \leq r$ ) in the (15)), we can assume that  $j_1 = 1$ . Thus,  $D_{11}$  is a nonsingular  $m_1 \times m_1$  matrix with  $m_1 = n_{\sigma_1(1)}$  and  $D_{j1} = \mathbf{0}$  ( $2 \leq j \leq r$ ). Therefore, the first group of components of the mapping  $h$  is independent of the variables  $\zeta_{\sigma_1(2)}, \dots, \zeta_{\sigma_1(r)}$ . For the simplicity of notation, we assume that  $\sigma_1$  is the identity permutation, i.e.  $\sigma_1(i) = i$  ( $1 \leq i \leq r$ ).

Next, let

$$Pr_1\Sigma(\mathbf{n}; \mathbf{p}) := \Sigma(\mathbf{n}; \mathbf{p}) \cap \{\zeta_1 = 0\}, \quad Pr_1\Sigma(\mathbf{m}; \mathbf{q}) := \Sigma(\mathbf{m}; \mathbf{q}) \cap \{\xi_1 = 0\}.$$

Since the first group of components of  $h$  is independent of  $\zeta_2, \dots, \zeta_r$ , we can consider the restriction  $\tilde{h}$  of  $h$  to  $\Sigma(\mathbf{n}; \mathbf{p}) \cap \{\zeta_1 = 0\} =: Pr_1\Sigma(\mathbf{n}; \mathbf{p})$  as follows:

$$\tilde{h} : Pr_1\Sigma(\mathbf{n}; \mathbf{p}) \rightarrow Pr_1\Sigma(\mathbf{m}; \mathbf{q})$$

$$\tilde{h}(\zeta_2, \dots, \zeta_r) = (\zeta_2, \dots, \zeta_r) \begin{pmatrix} D_{22} & D_{23} & \cdots & D_{2r} \\ D_{32} & D_{33} & \cdots & D_{3r} \\ \dots & \dots & \dots & \dots \\ D_{r2} & D_{r3} & \cdots & D_{rr} \end{pmatrix}.$$

Thus  $\tilde{h}$  is a biholomorphic linear mapping between  $Pr_1\Sigma(\mathbf{n}; \mathbf{p})$  and  $Pr_1\Sigma(\mathbf{m}; \mathbf{q})$ . By the same reasoning as above, we get that, after a permutation  $\sigma_2$  of the index of  $\zeta_i$  ( $2 \leq i \leq r$ ),  $D_{22}$  is a nonsingular  $m_2 \times m_2$  matrix with  $m_2 = n_{\sigma_2(2)}$  and  $D_{j2} = \mathbf{0}$  for  $3 \leq j \leq r$ . Again, for the simplicity of notation, we assume that  $\sigma_2$  is the identity permutation.

In the same way we can show that for each  $i = 1, \dots, r$ , after a permutation  $\sigma_i$  of the index of  $\zeta_j$  ( $i \leq j \leq r$ ),  $D_{ii}$  is a nonsingular  $m_i \times m_i$  matrix with  $m_i = n_{\sigma_i(i)}$  and  $D_{jk} = \mathbf{0}$  for  $k < j \leq r$ . Thus, if we let  $\sigma = \sigma_r \circ \dots \circ \sigma_1$ , then we have

$$h(\zeta_1, \zeta_2, \dots, \zeta_r) = (\zeta_{\sigma(1)}, \zeta_{\sigma(2)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1r} \\ \mathbf{0} & D_{22} & \cdots & D_{2r} \\ \dots & \dots & \ddots & \dots \\ \mathbf{0} & \mathbf{0} & \cdots & D_{rr} \end{pmatrix}.$$

Now we prove that  $D_{ij} = 0$  ( $1 \leq i < j \leq r$ ). In fact, we will show that for  $1 \leq i \leq r$ , the  $i$ -th column of the above matrix of  $h$  has only one nonzero block. Since every block  $D_{ii}$  ( $1 \leq i \leq r$ ) is nonsingular, we get that all other blocks  $D_{ij} = 0$  ( $1 \leq i < j \leq r$ ).

Suppose that there exist at least two nonzero blocks on some column, say the last column, of the above matrix of  $h$ . Then the system of linear equations

$$\sum_{j=1}^r \zeta_{\sigma(j)} D_{jr} = 0$$

has a solution  $(\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \dots, \gamma_{\sigma(r)}) \in (\mathbb{C}^{n_{\sigma(1)}} \setminus \{0\}) \times (\mathbb{C}^{n_{\sigma(2)}} \setminus \{0\}) \times \dots \times (\mathbb{C}^{n_{\sigma(r)}} \setminus \{0\})$  such that  $\sum_{j=1}^r \|\gamma_j\|^{2p_j} = \sum_{j=1}^r \|\gamma_{\sigma(j)}\|^{2p_{\sigma(j)}} = 1$ . That is  $(\gamma_1, \dots, \gamma_r) \in b_0\Sigma(\mathbf{n}, \mathbf{p})$ , but  $h(\gamma_1, \dots, \gamma_r) \notin b_0\Sigma(\mathbf{m}; \mathbf{q})$ . This is a contradiction with (13). Thus, each column of the above matrix of  $h$  has only one nonzero block and  $D_{ij} = 0$ ,  $1 \leq i < j \leq r$ . That is,

$$h(\zeta_1, \zeta_2, \dots, \zeta_r) = (\zeta_{\sigma(1)}, \zeta_{\sigma(2)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} D_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & D_{22} & \cdots & \mathbf{0} \\ \dots & \dots & \ddots & \dots \\ \mathbf{0} & \mathbf{0} & \cdots & D_{rr} \end{pmatrix}.$$

For each fixed  $j$  ( $1 \leq j \leq r$ ), if  $\|\zeta_{\sigma(j)}\|^2 < 1$ , then  $(\underbrace{0, \dots, 0}_{\sigma(j)-1}, \zeta_{\sigma(j)}, 0, \dots, 0) \in \Sigma(\mathbf{n}, \mathbf{p})$  and  $h(\underbrace{0, \dots, 0}_{\sigma(j)-1}, \zeta_{\sigma(j)}, 0, \dots, 0) = (\underbrace{0, \dots, 0}_{j-1}, \zeta_{\sigma(j)}, 0, \dots, 0) \text{diag}(D_{11}, \dots, D_{rr}) = (0, 0, \dots, 0, \zeta_{\sigma(j)} D_{jj}, 0, \dots, 0) \in \Sigma(\mathbf{m}, \mathbf{q})$ . Thus  $\|\zeta_{\sigma(j)} D_{jj}\|^2 < 1$ . On the other hand, for  $\|\xi_{\sigma^{-1}(j)}\|^2 < 1$ , then  $\|\xi_{\sigma^{-1}(j)} D_{jj}^{-1}\|^2 < 1$ . This indicates that  $D_{jj}(\mathbf{B}^{n_{\sigma(j)}}) \subset \mathbf{B}^{n_{\sigma(j)}}$  and  $D_{jj}^{-1}(\mathbf{B}^{n_{\sigma(j)}}) \subset \mathbf{B}^{n_{\sigma(j)}}$ . Therefore,  $D_{jj}$  is a unity transformation of  $\mathbb{C}^{n_{\sigma(j)}}$  for  $1 \leq j \leq r$ .

For  $2 \leq i \leq r$  and any  $\zeta_{\sigma(1)} \in \mathbb{C}^{n_{\sigma(1)}}$  with  $\|\zeta_{\sigma(1)}\|^2 < 1$ , there exists  $\zeta_{\sigma(i)} \in \mathbb{C}^{n_{\sigma(i)}}$  such that  $(0, \dots, 0, \zeta_{\sigma(1)}, 0, \dots, 0, \zeta_{\sigma(i)}, 0, \dots, 0) \in b\Sigma(\mathbf{n}; \mathbf{p})$ . Thus,  $(\zeta_{\sigma(1)} D_{11}, 0, \dots, 0, \zeta_{\sigma(i)} D_{ii}, 0, \dots, 0) \in b\Sigma(\mathbf{m}; \mathbf{q})$ . That is, from  $\|\zeta_{\sigma(1)}\|^{2p_{\sigma(1)}} + \|\zeta_{\sigma(i)}\|^{2p_{\sigma(i)}} = 1$ , we can get

$$\|\zeta_{\sigma(1)}\|^{2q_1} + \|\zeta_{\sigma(i)}\|^{2q_i} (= \|\zeta_{\sigma(1)} D_{11}\|^{2q_1} + \|\zeta_{\sigma(i)} D_{ii}\|^{2q_i}) = 1.$$

Hence,  $p_{\sigma(i)} = q_i$ ,  $1 \leq i \leq r$ . This finish the proof of Lemma 2.7 in Case (i).

**Case (ii).** In this case, without loss of generality (note that  $h : \Sigma(\mathbf{n}; \mathbf{p}) \rightarrow \Sigma(\mathbf{m}; \mathbf{q})$  is a biholomorphic linear isomorphism), we can assume  $q_1 = 1$ , and then  $\varepsilon = 1$ . We will prove  $p_1 = 1$  here.

Since  $b_1\Sigma(\mathbf{n}; \mathbf{p}) = \bigcup_{j=1+\delta}^r bPr_j\Sigma(\mathbf{n}, \mathbf{p})$  and  $b_1\Sigma(\mathbf{m}; \mathbf{q}) = \bigcup_{j=2}^r bPr_j\Sigma(\mathbf{m}; \mathbf{q})$ , where  $Pr_j\Sigma(\mathbf{n}, \mathbf{p}) := \Sigma(\mathbf{n}, \mathbf{p}) \cap \{\zeta_j = 0\}$  and  $Pr_j\Sigma(\mathbf{m}; \mathbf{q}) := \Sigma(\mathbf{m}; \mathbf{q}) \cap \{\xi_j = 0\}$ , by (14), we have

$$h\left(\bigcup_{j=1+\delta}^r bPr_j\Sigma(\mathbf{n}, \mathbf{p})\right) \subset \bigcup_{j=2}^r bPr_j\Sigma(\mathbf{m}; \mathbf{q}).$$

By the same argument in the proof of Lemma 2.6 above, we can get

$$h(\mathbf{B}^{n_1} \times \{0\} \times \cdots \times \{0\}) \subset \mathbf{B}^{m_1} \times \{0\} \times \cdots \times \{0\}.$$

Thus we have  $D_{1j} = \mathbf{0}$  ( $2 \leq j \leq r$ ) for the matrix of  $h$ . Apply the same argument to  $h^{-1}$ , we get

$$h^{-1}(\mathbf{B}^{m_1} \times \{0\} \times \cdots \times \{0\}) \subset \mathbf{B}^{n_1} \times \{0\} \times \cdots \times \{0\}.$$

Thus  $h|_{\mathbf{B}^{n_1} \times \{0\} \times \cdots \times \{0\}}$  is a biholomorphism between  $\mathbf{B}^{n_1} \times \{0\} \times \cdots \times \{0\}$  and  $\mathbf{B}^{m_1} \times \{0\} \times \cdots \times \{0\}$ .

In particular, we get that  $n_1 = m_1$ .

Since  $h$  is a holomorphic linear isomorphism of  $\mathbb{C}^{|\mathbf{n}|}$  onto  $\mathbb{C}^{|\mathbf{m}|}$  and  $D_{1j} = 0$  ( $2 \leq j \leq r$ ), we obtain that  $D_{11}$  and  $\begin{pmatrix} D_{22} & \cdots & D_{2r} \\ \cdots & \cdots & \cdots \\ D_{r2} & \cdots & D_{rr} \end{pmatrix}$  are invertible constant matrices. Moreover, we have

$$h^{-1}(\xi_1, \dots, \xi_r) = (\xi_1, \dots, \xi_r) \begin{pmatrix} D_{11}^{-1} & 0 & \cdots & 0 \\ E_{21} & E_{22} & \cdots & E_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ E_{r1} & E_{r2} & \cdots & E_{rr} \end{pmatrix}.$$

If  $\sum_{j=2}^r \|\zeta_j\|^{2p_j} < 1$ , then  $(0, \zeta_2, \dots, \zeta_r) \in \Sigma(\mathbf{n}, \mathbf{p})$  and

$$h(0, \zeta_2, \dots, \zeta_r) = \left( \sum_{j=2}^r \zeta_j D_{j1}, \sum_{j=2}^r \zeta_j D_{j2}, \dots, \sum_{j=2}^r \zeta_j D_{jr} \right) \in \Sigma(\mathbf{m}, \mathbf{q}).$$

Thus

$$\sum_{k=2}^r \left\| \sum_{j=2}^r \zeta_j D_{jk} \right\|^{2q_k} < 1 - \left\| \sum_{j=2}^r \zeta_j D_{j1} \right\|^{2q_1} \leq 1.$$

By the same way, for  $\sum_{j=2}^r \|\xi_j\|^{2q_j} < 1$ , we have

$$\sum_{k=2}^r \left\| \sum_{j=2}^r \xi_j E_{jk} \right\|^{2p_k} < 1 - \left\| \sum_{j=2}^r \xi_j E_{j1} \right\|^2 \leq 1.$$

This indicates that the mapping

$$\tilde{h} : Pr_1\Sigma(\mathbf{n}; \mathbf{p}) \rightarrow Pr_1\Sigma(\mathbf{m}; \mathbf{q})$$

$$\tilde{h}(\zeta_2, \dots, \zeta_r) = (\zeta_2, \dots, \zeta_r) \begin{pmatrix} D_{22} & \cdots & D_{2r} \\ \cdots & \cdots & \cdots \\ D_{r2} & \cdots & D_{rr} \end{pmatrix}$$

is a biholomorphic linear mapping between  $Pr_1\Sigma(\mathbf{n}; \mathbf{p})$  and  $Pr_1\Sigma(\mathbf{m}; \mathbf{q})$ . Since  $p_2 \neq 1$  and  $q_2 \neq 1$ , we can apply the conclusion in the case (i) to get that there exists a permutation  $\sigma$  ( $\in S_{r-1}$ ) of  $\{2, \dots, r\}$  such that  $n_{\sigma(i)} = m_i$ ,  $p_{\sigma(i)} = q_i$  ( $2 \leq i \leq r$ ) and

$$\tilde{h}(\zeta_2, \dots, \zeta_r) = (\zeta_{\sigma(2)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} D_{22} & & \\ & \ddots & \\ & & D_{rr} \end{pmatrix},$$

where  $D_{ii}$  is a unitary transformation of  $\mathbb{C}^{m_i}$  ( $m_i = n_{\sigma(i)}$ ) for  $2 \leq i \leq r$ . Therefore,

$$h(\zeta_1, \zeta_2, \dots, \zeta_r) = (\zeta_1, \zeta_{\sigma(2)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} D_{11} & & & \\ D_{21} & D_{22} & & \\ \vdots & & \ddots & \\ D_{r1} & & & D_{rr} \end{pmatrix}.$$

If  $\|\zeta_1\|^2 < 1$ , then  $(\zeta_1, 0, \dots, 0) \in \Sigma(\mathbf{n}, \mathbf{p})$  and  $h(\zeta_1, 0, \dots, 0) = (\zeta_1 D_{11}, 0, \dots, 0) \in \Sigma(\mathbf{m}, \mathbf{q})$ . Thus  $\|\zeta_1 D_{11}\|^2 < 1$ . On the other hand, for  $\|\xi_1\|^2 < 1$ , then  $\|\xi_1 D_{11}^{-1}\|^2 < 1$ . This indicates that  $D_{11}(\mathbf{B}^{n_1}) \subset \mathbf{B}^{n_1}$  and  $D_{11}^{-1}(\mathbf{B}^{n_1}) \subset \mathbf{B}^{n_1}$ . Therefore,  $D_{11}$  is a unitary transformation of  $\mathbb{C}^{n_1}$ .

If  $\|\zeta_{\sigma(j)}\|^2 = 1$ , then  $(0, \dots, 0, \zeta_{\sigma(j)}, 0, \dots, 0) \in b\Sigma(\mathbf{n}; \mathbf{p})$ , and then

$$h(0, \dots, 0, \zeta_{\sigma(j)}, 0, \dots, 0) = (\zeta_{\sigma(j)} D_{j1}, 0, \dots, 0, \zeta_{\sigma(j)} D_{jj}, 0, \dots, 0) \in b\Sigma(\mathbf{m}; \mathbf{q}).$$

Thus

$$\|\zeta_{\sigma(j)} D_{j1}\|^2 = 1 - \|\zeta_{\sigma(j)} D_{jj}\|^{2q_j} = 1 - \|\zeta_{\sigma(j)}\|^{2q_j} = 0$$

(Note  $D_{jj}$  is a unitary matrix). Hence  $D_{j1} = \mathbf{0}$  for  $2 \leq j \leq r$ . Thus we get

$$h(\zeta_1, \zeta_2, \dots, \zeta_r) = (\zeta_1, \zeta_{\sigma(2)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} D_{11} & & & \\ & D_{22} & & \\ & & \ddots & \\ & & & D_{rr} \end{pmatrix},$$

where  $D_{ii}$  are unitary transformations of  $\mathbb{C}^{n_i}$  for  $1 \leq i \leq r$ .

Take  $\zeta_1 \in \mathbb{C}^{n_1}$ ,  $\zeta_{\sigma(2)} \in \mathbb{C}^{n_{\sigma(2)}}$  (note  $\sigma$  is a permutation of  $\{2, \dots, r\}$ ) such that  $\|\zeta_1\|^{2p_1} = \frac{1}{2}$  and  $\|\zeta_{\sigma(2)}\|^{2p_{\sigma(2)}} = \frac{1}{2}$ . Then

$$h(\underbrace{\zeta_1, 0, \dots, 0}_{\sigma(2)-1}, \zeta_{\sigma(2)}, 0, \dots, 0) = (\zeta_1 D_{11}, \zeta_{\sigma(2)} D_{22}, 0, \dots, 0) \in b\Sigma(\mathbf{m}, \mathbf{q}).$$

Hence

$$\|\zeta_1\|^2 = \|\zeta_1 D_{11}\|^2 = 1 - \|\zeta_{\sigma(2)} D_{22}\|^{2q_2} = 1 - \|\zeta_{\sigma(2)}\|^{2q_2} = 1 - \|\zeta_{\sigma(2)}\|^{2p_{\sigma(2)}} = \frac{1}{2}$$

(Note  $q_1 = 1$  here). So we have  $p_1 = 1$  ( $= q_1$ ). The proof of Lemma 2.8 is finished.

Remark. Lemma 2.8 is an extension of Theorem 1.A to the case of the holomorphic linear isomorphisms between two equidimensional generalized complex ellipsoids.

### 3 Proof of the Main Theorem

#### Proof of Theorem 1.1

Let  $f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  be a biholomorphism. By Lemma 2.6, we have  $f(\Omega_1 \times \{0\}) \subset \Omega_2 \times \{0\}$ . In particular, we have  $f(0, 0) \in \Omega_2 \times \{0\}$ . Thus, we can choose an automorphism  $\Phi \in \Gamma(H_{\Omega_2}(\mathbf{m}, \mathbf{q}))$  such that  $\Phi \circ f(0, 0) = (0, 0)$ . Thus  $\Phi \circ f : H_{\Omega_1}(\mathbf{n}; \mathbf{p}) \rightarrow H_{\Omega_2}(\mathbf{m}; \mathbf{q})$  is a biholomorphism with  $\Phi \circ f(0, 0) = (0, 0)$ . Since any Hua domain is a bounded circular domain and contains the origin, by Cartan's theorem,  $g(\mu, \zeta_1, \dots, \zeta_r) = \Phi \circ f(\mu, \zeta_1, \dots, \zeta_r)$  is a biholomorphic linear mapping between  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  and  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$ , namely

$$g(z, w_{(1)}, \dots, w_{(r)}) = (z, w_{(1)}, \dots, w_{(r)}) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (z, w_{(1)}, \dots, w_{(r)}) \begin{pmatrix} A & B_1 & \dots & B_r \\ C_1 & D_{11} & \dots & D_{1r} \\ \dots & \dots & \dots & \dots \\ C_r & D_{r1} & \dots & D_{rr} \end{pmatrix},$$

where  $A, B, C, D$  are constant matrices. By Lemma 2.6, we have  $g(\Omega_1 \times \{0\}) = \Omega_2 \times \{0\}$ , and this means  $B = 0$ .

Let  $|\mathbf{n}| = n_1 + \cdots + n_r, |\mathbf{m}| = m_1 + \cdots + m_r$ . Let  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$  be two generalized complex ellipsoids defined respectively by

$$\begin{aligned}\Sigma(\mathbf{n}; \mathbf{p}) &:= \left\{ (\zeta_1, \dots, \zeta_r) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} : \sum_{j=1}^r \|\zeta_j\|^{2p_j} < 1 \right\}, \\ \Sigma(\mathbf{m}; \mathbf{q}) &:= \left\{ (\xi_1, \dots, \xi_r) \in \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_r} : \sum_{j=1}^r \|\xi_j\|^{2q_j} < 1 \right\}.\end{aligned}$$

Since  $g$  is a holomorphic linear isomorphism of  $\mathbb{C}^{d_1+|\mathbf{n}|}$  onto  $\mathbb{C}^{d_2+|\mathbf{m}|}$  and  $B = 0$ , we obtain that  $A$  and  $D$  are invertible constant matrices, moreover

$$\begin{aligned}g^{-1}(z', w'_{(1)}, \dots, w'_{(r)}) &= (z', w'_{(1)}, \dots, w'_{(r)}) \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix} \\ &= (z', w'_{(1)}, \dots, w'_{(r)}) \begin{pmatrix} A^{-1} & 0 & \cdots & 0 \\ E_1 & F_{11} & \cdots & F_{1r} \\ \cdots & \cdots & \cdots & \cdots \\ E_r & F_{r1} & \cdots & F_{rr} \end{pmatrix}.\end{aligned}$$

Note  $g(z, w_{(1)}, \dots, w_{(r)}) = \Phi \circ f(z, w_{(1)}, \dots, w_{(r)})$  is a holomorphically linear isomorphism of  $H_{\Omega_1}(\mathbf{n}; \mathbf{p})$  to  $H_{\Omega_2}(\mathbf{m}; \mathbf{q})$ . If  $(\zeta_1, \dots, \zeta_r) \in \Sigma(\mathbf{n}; \mathbf{p})$ , that is,

$$\sum_{j=1}^r \|\zeta_j\|^{2p_j} < 1 \quad (= N_{\Omega_1}(0, 0)),$$

then  $(0, \zeta_1, \dots, \zeta_r) \in H_{\Omega_1}(\mathbf{n}, \mathbf{p})$  and

$$g(0, \zeta_1, \dots, \zeta_r) = \left( \sum_{j=1}^r \zeta_j C_j, \sum_{j=1}^r \zeta_j D_{j1}, \dots, \sum_{j=1}^r \zeta_j D_{jr} \right) \in H_{\Omega_2}(\mathbf{m}, \mathbf{q}).$$

Thus, by Proposition 2.1(b), we obtain

$$\sum_{j=1}^r \left\| \sum_{i=1}^r \zeta_i D_{ij} \right\|^{2q_j} < N_{\Omega_2} \left( \sum_{j=1}^r \zeta_j C_j, \sum_{j=1}^r \overline{\zeta_j C_j} \right) \leq 1.$$

By the same way, if  $(\xi_1, \dots, \xi_r) \in \Sigma(\mathbf{m}; \mathbf{q})$ , then

$$\sum_{j=1}^r \left\| \sum_{i=1}^r \xi_i F_{ij} \right\|^{2p_j} < N_{\Omega_1} \left( \sum_{j=1}^r \xi_j E_j, \sum_{j=1}^r \overline{\xi_j E_j} \right) \leq 1.$$

This indicates that the mapping

$$h : \Sigma(\mathbf{n}; \mathbf{p}) \rightarrow \Sigma(\mathbf{m}; \mathbf{q})$$

$$h(\zeta_1, \dots, \zeta_r) := (\zeta_1, \dots, \zeta_r) \begin{pmatrix} D_{11} & \cdots & D_{1r} \\ \cdots & \cdots & \cdots \\ D_{r1} & \cdots & D_{rr} \end{pmatrix}$$

is a biholomorphic linear mapping between  $\Sigma(\mathbf{n}; \mathbf{p})$  and  $\Sigma(\mathbf{m}; \mathbf{q})$ . By Lemma 2.8, we get

$$h(\zeta_1, \dots, \zeta_r) = (\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} D_{11} & & \\ & \ddots & \\ & & D_{rr} \end{pmatrix},$$

where  $\sigma \in S_r$  is a permutation such that  $n_{\sigma(i)} = m_i$ ,  $p_{\sigma(i)} = q_i$  for  $1 \leq i \leq r$ , and  $D_{ii}$  is a unitary transformation of  $\mathbb{C}^{m_i}(n_{\sigma(i)} = m_i)$  for  $1 \leq i \leq r$ .

Now we prove  $C = 0$ . Thus the matrix of  $g = \Phi \circ f$  is a block diagonal matrix.

Since  $g(bH_{\Omega_1}(\mathbf{n}; \mathbf{p})) = bH_{\Omega_2}(\mathbf{m}; \mathbf{q})$ , we have that if

$$\|\zeta_{\sigma(j)}\|^2 = N_{\Omega_1}(0, 0)^{\frac{1}{p_{\sigma(j)}}} (= 1)$$

(that is,  $(\underbrace{0, \dots, 0}_{\sigma(j)}, \zeta_{\sigma(j)}, 0, \dots, 0) \in bH_{\Omega_1}(\mathbf{n}, \mathbf{p}))$ , then

$$g(\underbrace{0, \dots, 0}_{\sigma(j)}, \zeta_{\sigma(j)}, 0, \dots, 0) = (\zeta_{\sigma(j)} C_j, 0, \dots, 0, \zeta_{\sigma(j)} D_{jj}, 0, \dots, 0) \in bH_{\Omega_2}(\mathbf{m}, \mathbf{q}).$$

Since  $D_{jj}$  is a unitary matrix, we have

$$\|\zeta_{\sigma(j)} D_{jj}\|^2 = N_{\Omega_2}(\zeta_{\sigma(j)} C_j, \overline{\zeta_{\sigma(j)} C_j})^{\frac{1}{q_j}} (= \|\zeta_{\sigma(j)}\|^2 = 1).$$

By Proposition 2.1(b), we have  $\zeta_{\sigma(j)} C_j = 0$  for all  $\|\zeta_{\sigma(j)}\| = 1$ . Thus  $C_j = 0$  ( $1 \leq j \leq r$ ).

Therefore

$$g(z, w_{(1)}, \dots, \zeta_{(r)}) = (z, w_{(\sigma(1))}, \dots, w_{(\sigma(r))}) \begin{pmatrix} A & & & \\ & D_{11} & & \\ & & \ddots & \\ & & & D_{rr} \end{pmatrix},$$

where  $A$  is a holomorphically linear isomorphism of  $\Omega_1$  onto  $\Omega_2$ ,  $\sigma \in S_r$  is a permutation such that  $n_{\sigma(i)} = m_i$ ,  $p_{\sigma(i)} = q_i$  for  $1 \leq i \leq r$  and  $D_{ii}$  is a unitary transformation of  $\mathbb{C}^{m_i}(n_{\sigma(i)} = m_i)$  for  $1 \leq i \leq r$ . The proof of Theorem 1.1 is completed.

### Proof of Corollary 1.2

Since  $\Gamma(H_{\Omega}(\mathbf{n}; \mathbf{p}))$  is a subgroup of  $\text{Aut}(H_{\Omega}(\mathbf{n}; \mathbf{p}))$ , Theorem 1.1 immediately implies Corollary 1.2. This proves Corollary 1.2.

### Proof of Theorem 1.3

Since  $f : H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \rightarrow H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)$  is a proper holomorphic mapping between two equidimensional Hua domains, by Proposition 2.3,  $f$  extends holomorphically to a neighborhood  $V$  of  $\overline{H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)}$  with

$$f(bH_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)) \subset bH_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2).$$

Define

$$S := \{\xi \in V : J_f(\xi) = 0\},$$

where  $J_f(\xi) = \det(\frac{\partial f_i}{\partial \xi_j})(\xi)$  is the holomorphic Jacobian determinant of

$$f(\xi) = (f_1(\xi), \dots, f_{d+|\mathbf{n}|}(\xi)) \quad (\xi \in V).$$

By Proposition 2.4 and Lemma 2.5, we have

$$f(S \cap b_0 H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)) \subset (b\Omega_2 \times \{0\}) \cup b_1 H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2). \quad (17)$$

If  $S \cap H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \neq \emptyset$ , then, from the assumption that the subset  $(b\Omega_1 \times \{0\}) \cup b_1 H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)$  of  $bH_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)$  is contained in some complex analytic set of complex codimension at least 2, we have  $S \cap b_0 H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \neq \emptyset$ . Take an irreducible component  $S'$  of  $S$  with  $S' \cap H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \neq \emptyset$ . Then, from Proposition 2.4 (a), the intersection  $E_{S'}$  of  $S'$  with  $b_0 H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)$  is a real analytic submanifold of real dimension  $2(d_1 + |\mathbf{n}_1|) - 3$  on a dense, open subset of  $E_{S'}$  (Otherwise,  $S'$  cannot

be separated by  $E_{S'}$  (e.g., see Rudin [23], Theorem 14.4.5) and thus  $S'$  cannot be separated by  $bH_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)$ . This is impossible). From (17), we also have

$$f(E_{S'}) \subset (b\Omega_2 \times \{0\}) \cup b_1H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2).$$

Thus, by the uniqueness theorem,

$$f(S' \cap H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1)) \subset \bigcup_{j=1+\varepsilon}^r Pr_j(H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)), \quad (18)$$

where  $Pr_j(H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)) := H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2) \cap \{w'_{(j)} = 0\}$  for  $1 + \varepsilon \leq j \leq r$  and

$$\varepsilon = \begin{cases} 1 & \text{if the first component } p_2^1 \text{ of } \mathbf{p}_2 \text{ equals to 1,} \\ 0 & \text{if the first component } p_2^1 \text{ of } \mathbf{p}_2 \text{ does not equal to 1.} \end{cases}$$

Since  $\text{codim} S' = 1$ ,  $\text{codim}(\bigcup_{j=1+\varepsilon}^r Pr_j(H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2))) = \min\{n_2^{1+\varepsilon}, \dots, n_2^r\} \geq 2$ , where  $\mathbf{n}_2 = (n_2^1, \dots, n_2^r)$ . Since  $f : H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \rightarrow H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)$  is proper, this is a contradiction with (18). This means  $S \cap H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) = \emptyset$ .

Thus  $f : H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \rightarrow H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)$  is unbranched. Since Hua domain is simply connected by Proposition 2.1(c), we get that  $f : H_{\Omega_1}(\mathbf{n}_1; \mathbf{p}_1) \rightarrow H_{\Omega_2}(\mathbf{n}_2; \mathbf{p}_2)$  is a biholomorphism. The proof of Theorem 1.3 is completed.

#### Proof of Corollary 1.4

From Theorem 1.3 and Corollary 1.2, we immediately get Corollary 1.4.

#### Proof of Corollary 1.5

From Theorem 1.3, Corollary 1.5 is obviously.

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